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The Asymptotic Expansion of Bessel Functions of Large Order

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THE ASYMPTOTIC EXPANSION OF BESSEL FUNCTIONS OF LARGE ORDER

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CONTENTS

New expansions are obtained for the functions $I_\nu(\nu z)$, $K_\nu(\nu z)$ and their derivatives in terms of elementary functions, and for the functions $J_\nu(\nu z)$, $Y_\nu(\nu z)$, $H_\nu^{(1)}(\nu z)$, $H_\nu^{(2)}(\nu z)$ and their derivatives in terms of Airy functions, which are uniformly valid with respect to z when $|\nu|$ is large. New series for the zeros and associated values are derived by reversion and used to determine the distribution of the zeros of functions of large order in the z-plane. Particular attention is paid to the complex zeros of $Y_n(z)$ and the Hankel functions when the order n is an integer or half an odd integer, and for this purpose some new asymptotic expansions of the Airy functions are derived. Tables are given of complex zeros of Airy functions and other quantities which facilitate the rapid calculation of the smaller complex zeros of $Y_n(z)$, $Y'_n(z)$, and the Hankel functions and their derivatives, when $2n$ is an integer, to an accuracy of three or four significant figures.

1. Introduction and summary

The theory of the preceding paper (Olver 1954) may be applied to Bessel's differential equation to produce asymptotic expansions of the functions $J_\nu(\nu z)$, $Y_\nu(\nu z)$, $H_\nu^{(1)}(\nu z)$, $H_\nu^{(2)}(\nu z)$ and their derivatives in terms of Airy functions, which are uniformly valid with respect to z when $|\nu|$ is large. In this paper these expansions are considered in detail, together with the expansions of the zeros and associated values which are obtained from them by reversion. Similar expansions for the functions (but not for the derivatives or zeros) have also been given in a recent paper by Cherry (1950).

Other forms of expansion of Bessel functions of large order are those of Debye and Meissel (see, for example, Watson 1944, pp. 235–248) and the present writer (1952), which are valid when $|z-1| \gg |\nu|^{-\frac{2}{3}}$, $|z-1| \ll |\nu|^{-\frac{2}{3}}$ and $|z-1| \ll |\nu|^{-\frac{2}{5}}$, respectively. Because of the property of uniformity there are no such restrictions on the expansions of this paper, which are, moreover, more powerful than the earlier forms, particularly near the transition points $z = \pm 1$. In consequence, the new expansions are of considerable computational importance, and to facilitate their application comprehensive tables of the coefficients have been prepared for real arguments and orders and will be published elsewhere.

The investigation of the zeros of functions of large order has been rather difficult with the existing expansions for the functions. The only explicit series which have been given are those of McMahon (Watson 1944, pp. 505–506; also Bickley & Miller 1945) for the very large zeros, and those of the present writer (1951, 1952) which give the expansion of real zeros of fixed enumeration in the form

$$n + \alpha_1 n^{\frac{1}{3}} + \alpha_2 n^{-\frac{1}{3}} + \dots, \tag{1·1}$$

where† $n \equiv \nu$ and $\alpha_1, \alpha_2, \dots$ are numerical constants. The values of these constants increase for the larger zeros, and expansions of this form are useful only for the smaller zeros. Other series could be obtained by reversion of Debye's expansions, but they would not be valid for indefinitely large orders if the enumeration is kept fixed.

The series given below do not have these disadvantages. The asymptotic expansion of a zero of fixed enumeration is evolved in the form

$$\nu p_0(\nu^{-\frac{2}{3}}a) + \nu^{-1} p_1(\nu^{-\frac{2}{3}}a) + \nu^{-3} p_2(\nu^{-\frac{2}{3}}a) + \dots, \tag{1·2}$$

where a is the corresponding Airy function zero, and the coefficients $p_0(\zeta), p_1(\zeta), \dots$, are transcendental functions of the argument $\zeta \equiv \nu^{-\frac{2}{3}}a$ which may be pretabulated. The expansion (1·2) is uniformly valid with respect to all the zeros and numerically it is very powerful; it is effectively a power series in ν^{-2} in which the early coefficients actually decrease in magnitude. In addition to numerical applications the new series are of theoretical value in determining the asymptotic distribution of complex zeros of Bessel functions of large orders, both real and complex, and lead to some new and interesting results. It is found, for example, that if n is a positive integer, then in the domain $|\arg z| \leqslant \pi$ the function $Y_n(nz)$ has, in addition to its infinite set of real positive zeros, two infinite strings of zeros asymptotically near to the negative real axis, together with $2n$ zeros asymptotically near to the boundary of an eye-shaped domain in the z-plane whose extreme points are at $z = \pm 1$.

For completeness an account is also given of the analogous asymptotic expansions of the modified Bessel functions $I_\nu(\nu z)$ and $K_\nu(\nu z)$. These expansions, which are in terms of exponential and not Airy functions, may be obtained from the Debye expansions for $J_\nu(\nu z)$ and $H_\nu^{(1)}(\nu z)$ for complex arguments and orders (Watson 1944, pp. 262–268), but in this paper they will be derived from the defining differential equation. The only previous account of this approach appears to be that of Lehmer (1945) with corrections by Miller (1945), which deals briefly with $I_n(z)$ for real arguments and orders. The modified Bessel functions are considered first because their asymptotic theory is simpler than that of the Bessel functions themselves.

The arrangement of the paper is as follows. The expansions of $I_n(nz)$, $K_n(nz)$ and their derivatives for large positive orders are given in § 2. The extension of these results to complex orders is considered in § 3.

† Here and elsewhere in this paper we use the convention that when the symbol n is used to denote the order of the Bessel functions it is real and positive.

In § 4 expansions are developed for $J_n(nz)$, $Y_n(nz)$, $H_n^{(1)}(nz)$ and $H_n^{(2)}(nz)$ when the order is large and positive, and in § 5 these results are extended to include complex orders. The coefficients in these expansions are defined by recurrence formulae, and explicit expressions for them are obtained in § 6. This section also contains expansions for the derivatives of the Bessel functions.

Zeros are investigated in §§ 7, 8, 9 and 10. The asymptotic expansions of §§ 4, 5 and 6 are reverted in § 7 to give series for the zeros of $J_\nu(z)$ and $J'_\nu(z)$ and the respective associated values of $J'_\nu(z)$ and $J_\nu(z)$. The corresponding expansions for $Y_n(z)$ and $Y'_n(z)$ are considered in § 8, with particular reference to the complex zeros of these functions when $2n$ is an integer, and a similar investigation of the zeros of the Hankel functions and their derivatives is made in § 9. In § 10 tables are given which facilitate rapid calculation of approximate values of the complex zeros of $Y_n(z)$, $Y'_n(z)$, the Hankel functions and their derivatives, and examples are given of their application.

Certain properties of the Airy functions Ai and Bi are frequently used in the paper. Since they do not all appear to be well known, particularly in the case of Bi, they have been collected together and given in the Appendix.

2. The expansions of $I_n(nz)$ and $K_n(nz)$ for large positive orders

The functions $z^{\frac{1}{2}}I_n(nz)$ and $z^{\frac{1}{2}}K_n(nz)$ satisfy the differential equation

$$\frac{d^2w}{dz^2} = \left\{\left(\frac{1+z^2}{z^2}\right)n^2 - \frac{1}{4z^2}\right\}w. \tag{2·1}$$

This equation has transition points at $z = 0, \pm i$. We shall suppose throughout this section that z is confined to the half-plane $|\arg z| < \frac{1}{2}\pi$; the results obtained may always be extended subsequently to other phase ranges not including the imaginary axis† with the aid of the continuation formulae

$$\left.\begin{aligned} I_\nu(z\,e^{m\pi i}) &= e^{m\nu\pi i}\,I_\nu(z), \\ K_\nu(z\,e^{m\pi i}) &= e^{-m\nu\pi i}\,K_\nu(z) - \pi i \sin m\nu\pi \,\operatorname{cosec}\nu\pi\, I_\nu(z), \end{aligned}\right\} \tag{2·2}$$

m being an arbitrary integer.

Following the procedure described in the preceding paper (§ 2, cases A and C), we make simultaneous changes of variables in (2·1) from w, z to W, ζ respectively, given by

$$\left(\frac{d\zeta}{dz}\right)^2 = \frac{1+z^2}{z^2}, \qquad W = \left(\frac{dz}{d\zeta}\right)^{-\frac{1}{2}} w = \left(\frac{1+z^2}{z^2}\right)^{\frac{1}{4}} w. \tag{2·3}$$

Then W satisfies the equation

$$\frac{d^2W}{d\zeta^2} = \{n^2 + f(\zeta)\}\,W, \tag{2·4}$$

where

$$f(\zeta) = -\frac{\dot{z}^2}{4z^2} + \dot{z}^{\frac{1}{2}}\frac{d^2}{d\zeta^2}(\dot{z}^{-\frac{1}{2}}), \qquad \dot{z} \equiv \frac{dz}{d\zeta}. \tag{2·5}$$

† Expansions which include the imaginary axis in their region of validity are of a different kind; they may be obtained from the results given later in §§ 4 and 5 by means of the relations

$$I_\nu(z) = e^{-\frac{1}{2}\nu\pi i}\,J_\nu(z e^{\frac{1}{2}\pi i}), \quad K_\nu(z) = \tfrac{1}{2}\pi i\, e^{\frac{1}{2}\nu\pi i}\,H_\nu^{(1)}(z e^{\frac{1}{2}\pi i}).$$

The first of equations (2·3) may be integrated to give

$$\zeta = \surd(1+z^2) + \ln \frac{z}{1+\surd(1+z^2)}, \tag{2·6}$$

it being convenient to take the arbitrary constant of integration equal to zero. This relation maps the half-plane $|\arg z| < \frac{1}{2}\pi$ conformally on the domain **D** comprising the half-plane $\mathscr{R}\zeta > 0$ and the half-strip $|\mathscr{I}\zeta| < \frac{1}{2}\pi$, $\mathscr{R}\zeta \leqslant 0$. Corresponding points of the transformation are indicated in figures 1 and 2.

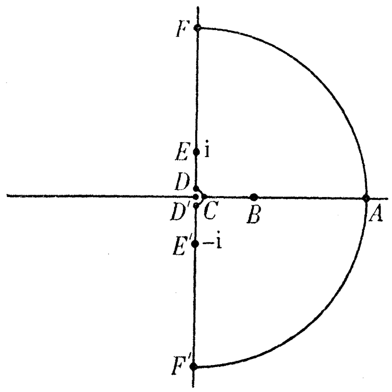

FIGURE 1. z-plane.

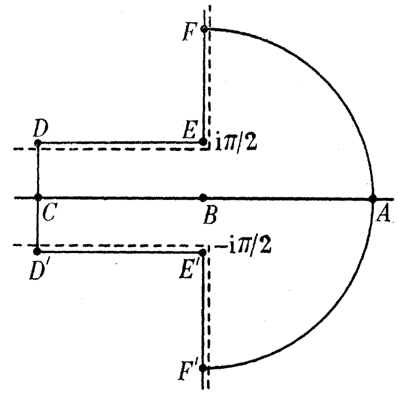

FIGURE 2. ζ-plane.

Substituting the first of (2·3) in (2·5) we find, after a little reduction, that

$$f(\zeta) = \frac{z^2(4-z^2)}{4(1+z^2)^3}. \tag{2·7}$$

Thus $f(\zeta)$ is a regular (holomorphic) function of ζ in **D**. Moreover, from (2·6) it is seen that $z \sim \zeta$ as $|\zeta| \to \infty$ in the right-hand half-plane, and if ζ lies in the strip $|\mathscr{I}\zeta| < \frac{1}{2}\pi$, $\mathscr{R}\zeta \leqslant 0$, then $z \sim 2\,e^{\zeta-1}$ as $|\zeta| \to \infty$. Hence

$$f(\zeta) = O(|\zeta|^{-2}) \tag{2·8}$$

as $|\zeta| \to \infty$ in **D**, uniformly with respect to $\arg \zeta$.

Thus the preliminary conditions are satisfied for the application of theorem A† to equation (2·4). Let **D′** denote the part of **D** bounded by the lines

$$\mathscr{I}\zeta = \pm(\tfrac{1}{2}\pi - \delta), \quad \mathscr{R}\zeta \leqslant \delta; \qquad \mathscr{R}\zeta = \delta, \quad |\mathscr{I}\zeta| \geqslant \tfrac{1}{2}\pi - \delta,$$

where $0 < \delta \leqslant \frac{1}{2}\pi$; these boundaries are indicated by the broken lines in figure 2. Then by taking a_1, a_2 to be the points at infinity on the negative and positive real axes respectively, the domains $\mathbf{D}_1$ and $\mathbf{D}_2$ of theorem A coincide with **D′** itself. Accordingly, if a sequence of coefficients‡ $\{U_s(\zeta)\}$ is defined by the equations $U_0(\zeta) = 1$ and

$$U_{s+1}(\zeta) = -\tfrac{1}{2}U_s'(\zeta) + \tfrac{1}{2}\int f(\zeta)\,U_s(\zeta)\,d\zeta \quad (s \geqslant 0), \tag{2·9}$$

then solutions $W_1(\zeta)$ and $W_2(\zeta)$ of (2·4) exist such that, if ζ lies in **D′**,

$$W_1(\zeta) \sim e^{n\zeta} \sum_{s=0}^{\infty} \frac{U_s(\zeta)}{n^s}, \qquad W_2(\zeta) \sim e^{-n\zeta} \sum_{s=0}^{\infty} (-)^s \frac{U_s(\zeta)}{n^s}, \tag{2·10}$$

as $n \to \infty$, uniformly with respect to ζ.

† Here and elsewhere the terms 'theorem A' and 'theorem B' refer to the theorems of the preceding paper, §5.

‡ The notation A_s of theorem A has been changed here to U_s for later convenience.

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We shall now express $I_n(nz)$ in terms of $W_1(\zeta)$ and $W_2(\zeta)$. From the second of (2·3) it follows that

$$I_n(nz) = \lambda_n(1+z^2)^{-\frac{1}{4}} W_1(\zeta) + \mu_n(1+z^2)^{-\frac{1}{4}} W_2(\zeta), \tag{2·11}$$

where λ_n, μ_n are independent of z. Because of the uniform property of the expansions (2·10), we may keep n fixed and let $z \to 0$ through positive values, or, equivalently, $\zeta \to -\infty$. It is then immediately obvious that $\mu_n = 0$ for all sufficiently large n. Next, substituting (2·10) in (2·11), we see that, for large n,

$$\lambda_n \sim e^{-n\zeta}(1+z^2)^{\frac{1}{4}} I_n(nz) \div \left\{ \sum_{s=0}^{\infty} \frac{U_s(\zeta)}{n^s} \right\}.$$

Now as $z \to +\infty$, n being fixed, we have from (2·6) and the well-known asymptotic expansion of $I_n(z)$ for large z,

$$\zeta = z + O(z^{-1}), \qquad I_n(nz) \sim (2\pi nz)^{-\frac{1}{2}} e^{nz},$$

and so

$$e^{-n\zeta}(1+z^2)^{\frac{1}{4}} I_n(nz) \to (2\pi n)^{-\frac{1}{2}}$$

as $z \to +\infty$. Hence if we now fix the arbitrary constants in (2·9) by making $U_{s+1}(+\infty) = 0$ $(s \geq 0)$, so that this relation becomes

$$U_{s+1}(\zeta) = -\tfrac{1}{2}U_s'(\zeta) - \tfrac{1}{2}\int_\zeta^\infty f(t)\, U_s(t)\, dt, \tag{2·12}$$

then λ_n can be replaced by $(2\pi n)^{-\frac{1}{2}}$ in equation (2·11) provided that the sign of equality is replaced by asymptotic equality. Thus we derive the desired result

$$I_n(nz) \sim \frac{e^{n\zeta}}{\sqrt{(2\pi n)}\,(1+z^2)^{\frac{1}{4}}} \sum_{s=0}^{\infty} \frac{U_s(\zeta)}{n^s} \tag{2·13}$$

as $n \to \infty$, uniformly with respect to z in the half-plane $|\arg z| \leq \frac{1}{2}\pi - \epsilon$, where ϵ is an arbitrary positive number in the range $0 < \epsilon < \frac{1}{2}\pi$.

The corresponding expansion for $K_n(nz)$ may be derived in a similar manner, using the relation

$$K_n(nz) \sim \left(\frac{\pi}{2nz}\right)^{\frac{1}{2}} e^{-nz}, \quad \text{as} \quad z \to +\infty.$$

The result is given by

$$K_n(nz) \sim \sqrt{\left(\frac{\pi}{2n}\right)} \frac{e^{-n\zeta}}{(1+z^2)^{\frac{1}{4}}} \sum_{s=0}^{\infty} (-)^s \frac{U_s(\zeta)}{n^s}, \tag{2·14}$$

as $n \to \infty$, uniformly with respect to z in $|\arg z| \leq \frac{1}{2}\pi - \epsilon$.

The linking of $I_n(nz)$ and $K_n(nz)$ with $W_1(\zeta)$ and $W_2(\zeta)$ may alternatively be effected by using the known behaviour of the functions at $z = 0$ in place of $z = +\infty$. This procedure leads to the result

$$U_s(-\infty) = \gamma_s, \tag{2·15}$$

where $\gamma_0, \gamma_1, \gamma_2, \ldots$ are the coefficients in the asymptotic expansion

$$\frac{\sqrt{(2\pi)}\, n^{n-\frac{1}{2}}}{e^n\, \Gamma(n)} \sim \gamma_0 + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2} + \ldots. \tag{2·16}$$

The first seven are given by

$$\left.\begin{aligned} &\gamma_0 = 1, \quad \gamma_1 = -\frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = \frac{139}{51840}, \\ &\gamma_4 = -\frac{571}{24\,88320}, \quad \gamma_5 = -\frac{1\,63879}{2090\,18880}, \quad \gamma_6 = \frac{52\,46819}{7\,52467\,96800}. \end{aligned}\right\} \tag{2·17}$$

The coefficients $U_s(\zeta)$ can be evaluated explicitly in terms of z. In order to arrive at a convenient practical form we introduce the auxiliary variable

$$u \equiv (1+z^2)^{-\frac{1}{2}}.$$

Then, using (2·3) and (2·7), we obtain

$$\frac{du}{d\zeta} = -u^2(1-u^2), \qquad f(\zeta) = \tfrac{1}{4}u^2(1-u^2)(5u^2-1), \tag{2·18}$$

and the recurrence relation (2·12) becomes

$$U_{s+1} = \tfrac{1}{2}u^2(1-u^2)\frac{dU_s}{du} + \tfrac{1}{8}\int_0^u (1-5u^2)\,U_s\,du. \tag{2·19}$$

From this result it is seen that U_s is a polynomial in u. For $s = 0, 1, 2, 3$, we have

$$\left.\begin{aligned} &U_0 = 1, \qquad U_1 = \tfrac{1}{8}u - \tfrac{5}{24}u^3, \qquad U_2 = \tfrac{9}{128}u^2 - \tfrac{77}{192}u^4 + \tfrac{385}{1152}u^6, \\ &U_3 = \tfrac{75}{1024}u^3 - \tfrac{4563}{5120}u^5 + \tfrac{17017}{9216}u^7 - \tfrac{85085}{82944}u^9. \end{aligned}\right\} \tag{2·20}$$

When $u = 1$ we see from (2·15) that $U_s = \gamma_s$, and this affords a useful check on the coefficients in (2·20).

The asymptotic expansions of the derivatives of the modified Bessel functions may be obtained by differentiating the expansions (2·13) and (2·14) with respect to z, the legitimacy of this process being a consequence of theorem A. Using (2·18) the expansions are found to be

$$I_n'(nz) \sim \frac{(1+z^2)^{\frac{1}{4}}}{z}\frac{e^{n\zeta}}{\sqrt{(2\pi n)}}\sum_{s=0}^{\infty}\frac{V_s}{n^s}, \qquad K_n'(nz) \sim -\sqrt{\left(\frac{\pi}{2n}\right)}\frac{(1+z^2)^{\frac{1}{4}}}{z}e^{-n\zeta}\sum_{s=0}^{\infty}(-)^s\frac{V_s}{n^s}, \tag{2·21}$$

where the coefficients V_s are polynomials in u, given by

$$V_s = U_s - u(1-u^2)\left(\tfrac{1}{2}U_{s-1} + u\frac{dU_{s-1}}{du}\right). \tag{2·22}$$

The first four are

$$\left.\begin{aligned} &V_0 = 1, \qquad V_1 = -\tfrac{3}{8}u + \tfrac{7}{24}u^3, \qquad V_2 = -\tfrac{15}{128}u^2 + \tfrac{99}{192}u^4 - \tfrac{455}{1152}u^6, \\ &V_3 = -\tfrac{105}{1024}u^3 + \tfrac{5577}{5120}u^5 - \tfrac{6545}{3072}u^7 + \tfrac{95095}{82944}u^9. \end{aligned}\right\} \tag{2·23}$$

They take the same values as the corresponding U_s at $u = 1$. Another check is given by

$$U_{2s}V_0 - U_{2s-1}V_1 + U_{2s-2}V_2 - \ldots + U_0V_{2s} = 0 \quad (s \geq 1), \tag{2·24}$$

which is derived from the Wronskian relation for $I_n(nz)$ and $K_n(nz)$.

3. The expansion of $I_\nu(\nu z)$ and $K_\nu(\nu z)$ for large complex orders

Let $\nu = n\,e^{i\vartheta}$, where ϑ is real and fixed and n is large and positive. The functions

$$(1+z^2)^{\frac{1}{4}}I_\nu(\nu z) \quad \text{and} \quad (1+z^2)^{\frac{1}{4}}K_\nu(\nu z)$$

satisfy the differential equation

$$\frac{d^2W}{d\zeta^2} = \{n^2 e^{2i\vartheta} + f(\zeta)\}\,W \tag{3·1}$$

(cf. (2·4)), where ζ is given by (2·6). With the substitution $\xi = \zeta\,e^{i\vartheta}$, this becomes

$$\frac{d^2W}{d\xi^2} = \{n^2 + e^{-2i\vartheta}f(\xi\,e^{-i\vartheta})\}\,W. \tag{3·2}$$

If z lies in the half-plane $|\arg z| < \frac{1}{2}\pi$, then ξ lies in a domain **E** which is the domain **D** of §2 rotated through an angle ϑ. Let **E**′ be the domain in the ξ-plane corresponding to **D**′ in the ζ-plane. By applying theorem A to (3·2), we find that solutions W_1, W_2 exist such that

$$W_1 \sim e^{n\xi} \sum_{s=0}^{\infty} \frac{u_s(\xi)}{n^s}, \qquad W_2 \sim e^{-n\xi} \sum_{s=0}^{\infty} (-)^s \frac{u_s(\xi)}{n^s}, \tag{3·3}$$

as $n \to \infty$, uniformly with respect to ξ in **E**′. The coefficients $u_s(\xi)$ are given by $u_0(\xi) = 1$ and

$$u_{s+1}(\xi) = -\tfrac{1}{2}u_s'(\xi) - \tfrac{1}{2}\int_{\xi}^{\infty e^{i\theta}} e^{-2i\vartheta} f(t\,e^{-i\vartheta})\, u_s(t)\, dt.$$

The function $(1+z^2)^{\frac{1}{4}} I_\nu(\nu z)$ may be expressed in terms of these solutions in a manner similar to that of §2, using the relations

$$I_\nu(\nu z) \sim \frac{(\frac{1}{2}\nu z)^\nu}{\Gamma(\nu+1)} \quad \text{as} \quad z \to 0, \qquad I_\nu(\nu z) \sim \frac{e^{\nu z}}{(2\pi\nu z)^{\frac{1}{2}}} \quad \text{as} \quad z \to +\infty. \tag{3·4}$$

The first of these is valid for all ν other than a negative integer, and the second holds when $|\arg \nu| < \frac{1}{2}\pi$. We find

$$I_\nu(\nu z) \sim \frac{e^{n\xi}}{\surd(2\pi\nu)\,(1+z^2)^{\frac{1}{4}}} \sum_{s=0}^{\infty} \frac{u_s(\xi)}{n^s}, \tag{3·5}$$

as $n \to \infty$, provided $|\vartheta| < \frac{1}{2}\pi$.

If the original variables ζ and ν are restored in place of n and ξ, it is seen that

$$u_s(\xi) = e^{-si\vartheta}\, U_s(\zeta),$$

where $U_s(\zeta)$ is defined by (2·12), and (3·5) becomes

$$I_\nu(\nu z) \sim \frac{e^{\nu\zeta}}{\surd(2\pi\nu)\,(1+z^2)^{\frac{1}{4}}} \sum_{s=0}^{\infty} \frac{U_s(\zeta)}{\nu^s}. \tag{3·6}$$

In other words, (2·13) *remains valid if n is replaced by the complex variable ν, provided that* $|\arg \nu| < \frac{1}{2}\pi$. *We may, in a similar manner, prove that the same is true of* (2·14).

When $\arg \nu$ lies outside the range $|\arg \nu| < \frac{1}{2}\pi$, the corresponding expansions may be deduced from (3·6) and the analogous expansion for $K_\nu(\nu z)$, by means of (2·2) and the formulae

$$I_{-\nu}(z) = I_\nu(z) + 2\pi^{-1} \sin \nu\pi K_\nu(z), \qquad K_{-\nu}(z) = K_\nu(z). \tag{3·7}$$

In general they are mixtures of both positive and negative exponential-type solutions.

4. The expansion of $J_n(nz)$, $Y_n(nz)$ and the Hankel functions for large positive orders

The functions $z^{\frac{1}{2}}J_n(nz)$ and $z^{\frac{1}{2}}Y_n(nz)$ satisfy

$$\frac{d^2w}{dz^2} = \left\{\left(\frac{1-z^2}{z^2}\right)n^2 - \frac{1}{4z^2}\right\} w. \tag{4·1}$$

This equation has transition points at $z = 0, \pm 1$. Taking $z_0 = 1$ and following the procedure described in the preceding paper (§2, case B), we introduce new variables W, ζ given by

$$\zeta\left(\frac{d\zeta}{dz}\right)^2 = \frac{1-z^2}{z^2}, \qquad W = \left(\frac{dz}{d\zeta}\right)^{-\frac{1}{2}} w. \tag{4·2}$$

Then W satisfies the equation

$$\frac{d^2W}{d\zeta^2} = \{n^2\zeta + f(\zeta)\} W, \tag{4·3}$$

where $$f(\zeta) = -\frac{\dot{z}^2}{4z^2} + \dot{z}^{\frac{1}{2}} \frac{d^2}{d\zeta^2}(\dot{z}^{-\frac{1}{2}}), \qquad \dot{z} \equiv \frac{dz}{d\zeta}. \tag{4·4}$$

From the first of (4·2) we obtain

$$\tfrac{2}{3}\zeta^{\frac{3}{2}} = \pm \int_1^z \frac{\sqrt{(1-z^2)}}{z}\, dz. \tag{4·5}$$

We adopt the lower sign here to make ζ real when z is real, and carrying out the integration we find that

$$\tfrac{2}{3}\zeta^{\frac{3}{2}} = \ln \frac{1+\sqrt{(1-z^2)}}{z} - \sqrt{(1-z^2)}. \tag{4·6}$$

In order to see how the z-plane is mapped on the ζ-plane it is helpful to introduce auxiliary variables σ and ρ, related to z and ζ by the equations

$$z = \operatorname{sech} \sigma, \qquad \tfrac{2}{3}\zeta^{\frac{3}{2}} = \rho = \sigma - \tanh \sigma. \tag{4·7}$$

The sector $0 < \arg z < \pi$ is mapped conformally on the half-strip $-\pi < \mathscr{I}\sigma < 0$, $\mathscr{R}\sigma > 0$, and on a domain in the ρ-plane comprising the half-plane $\mathscr{R}\rho < 0$ and the half-strip $-\pi < \mathscr{I}\rho < 0$, $\mathscr{R}\rho \geq 0$. This in turn is mapped conformally in the ζ-plane on a domain bounded by the real axis, the ray $\arg\{\zeta\, e^{\frac{1}{3}\pi i} - (\frac{3}{2}\pi)^{\frac{2}{3}}\} = 0$ and the curve whose parametric equation is given by

$$\zeta = (\tfrac{3}{2})^{\frac{2}{3}} (t - i\pi)^{\frac{2}{3}} \quad (0 \leq t < \infty). \tag{4·8}$$

Corresponding points of the transformation of the sector $|\arg z| < \pi$ are indicated in figures 3, 4, 5 and 6. For clarity only the parts of the σ and ρ domains corresponding to $0 < \arg z < \pi$ have been shown in figures 4 and 5. In figure 6 the points E, E' have affixes $(\frac{3}{2}\pi)^{\frac{2}{3}} e^{\mp\frac{1}{3}\pi i}$ and the curve ED has the equation (4·8), $E'D'$ being its conjugate.

For the purpose of applying theorem B† to (4·3) it is not sufficient to confine attention to the interior of the ζ-domain which corresponds to $|\arg z| < \pi$, because the distance between the boundary curves ED and $E'D'$ shrinks to zero as the curves approach infinity. It is necessary to investigate how the remainder of the ζ-plane is mapped on the z-plane. The first point to be noticed in this connexion is that the line BE of the ζ-plane is mapped on curves in the σ- and z-planes whose parametric equations are readily verified to be

$$\sigma = t - \tfrac{1}{2}\pi i \pm i \cos^{-1}\{(1 - t^{-1}\tanh t)^{\frac{1}{2}} \cosh t\}, \tag{4·9}$$

$$z = \pm (t \coth t - t^2)^{\frac{1}{2}} + i(t^2 - t \tanh t)^{\frac{1}{2}}, \tag{4·10}$$

where $t \equiv \mathscr{R}\sigma$ and describes the range $0 \leq t \leq t_0$, where $t_0 = 1·19968\ldots$ is the positive root of $t = \coth t$. These curves are indicated by the broken lines in figures 3 and 4; the points P, P' where they cut the imaginary z-axis (figure 3) have affixes

$$\pm i(t_0^2 - 1)^{\frac{1}{2}} = \pm i\, 0·66274\ldots$$

and correspond to the points $(\frac{3}{4}\pi)^{\frac{2}{3}} e^{\mp\frac{1}{3}\pi i}$ in the ζ-plane.

Next, we observe from (4·7) that the effect of increasing σ by $-i\pi$ is to increase ρ by $-i\pi$ and to change the sign of z.

† See first footnote on p. 331.

From these results we deduce that the sector $|\arg\zeta| < \frac{1}{3}\pi$ is mapped on the eye-shaped domain **K** in the z-plane bounded by the curve BPE, whose equation is (4·10), and its conjugate $BP'E'$. The transformation is not of course one-to-one and is more properly represented by

$$\tfrac{2}{3}\zeta^{\frac{3}{2}} = \ln\{1+\sqrt{(1-z^2)}\} - \mathrm{Ln}\, z - \sqrt{(1-z^2)} \qquad (4{\cdot}11)$$

(cf. (4·6)), where $\mathrm{Ln}\, z$ denotes the many-valued function whose imaginary part equals $\mathrm{i}\arg z$. If z lies in **K** and $m\pi < \arg z < (m+1)\pi$, where m is any integer, then ζ lies in the curved half-strip bounded by the curves

$$\zeta = (\tfrac{3}{2})^{\frac{2}{3}}(t - m\pi\mathrm{i})^{\frac{2}{3}}, \qquad \zeta = (\tfrac{3}{2})^{\frac{2}{3}}(t - \overline{m+1}\,\pi\mathrm{i})^{\frac{2}{3}} \quad (t \geq 0),$$

and the ray $\arg\zeta = -\frac{1}{3}\pi$ if $m \geq 0$, or $\arg\zeta = \frac{1}{3}\pi$ if $m < 0$.

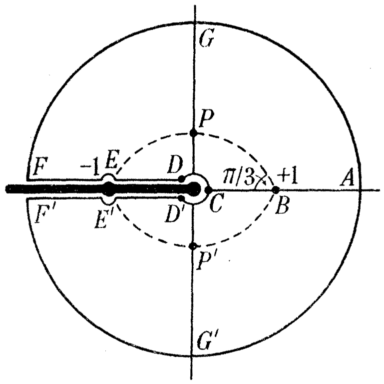

FIGURE 3. z-plane.

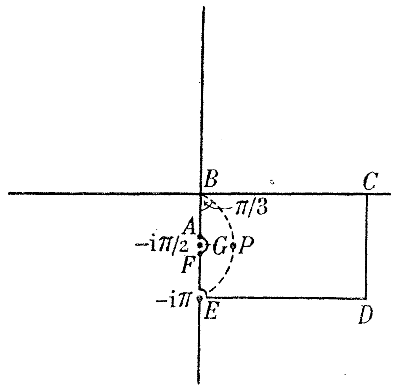

FIGURE 4. σ-plane.

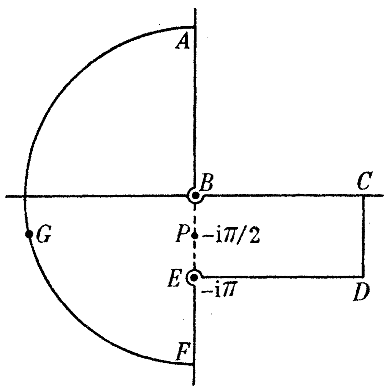

FIGURE 5. ρ-plane.

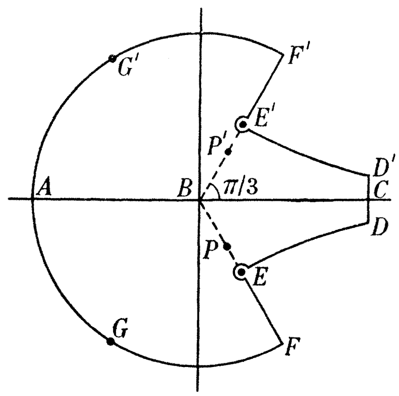

FIGURE 6. ζ-plane.

If z is regarded as a function of ζ it is now clear that $z(\zeta)$ is regular over the whole ζ-plane provided cuts are made along the rays $\arg\zeta = \pm\frac{1}{3}\pi$ from $\zeta = (\frac{3}{2}\pi)^{\frac{2}{3}}\mathrm{e}^{\pm\frac{1}{3}\pi\mathrm{i}}$ to infinity. We shall denote the open domain formed by the ζ-plane cut in this way by the symbol† **D**.

We return to the function $f(\zeta)$. Substituting the first of (4·2) in (4·4) and carrying out some reduction, we find that

$$f(\zeta) = \frac{5}{16\zeta^2} + \frac{\zeta}{4}\frac{z^2(z^2+4)}{(z^2-1)^3}. \qquad (4{\cdot}12)$$

The only point of **D** at which $z^2 = 1$ is $\zeta = 0$. Near this point, however, we find from (4·4) and the first of (4·2)

$$z(\zeta) = 1 - 2^{-\frac{1}{3}}\zeta + \tfrac{3}{10}2^{-\frac{2}{3}}\zeta^2 + \tfrac{1}{700}\zeta^3 + \zeta^4 O(1), \qquad f(\zeta) = \frac{2^{\frac{1}{3}}}{70} + \zeta O(1), \qquad (4{\cdot}13)$$

and so $f(\zeta)$ is a regular function of ζ throughout **D**.

† No confusion with the **D** of §§ 2 and 3 will arise.

If $|\zeta|\to\infty$ in the sector $|\arg(-\zeta)|<\frac{2}{3}\pi$, then $|z|\to\infty$ and from (4·6) and (4·12) we obtain

$$z=\tfrac{2}{3}(-\zeta)^{\frac{3}{2}}+\tfrac{1}{2}\pi+O(|\zeta|^{-\frac{3}{2}}),\qquad f(\zeta)\sim-\frac{1}{4\zeta^2}. \tag{4·14}$$

On the other hand, if $|\zeta|\to\infty$ in $|\arg\zeta|<\frac{1}{3}\pi$ then $|z|\to 0$, and from (4·11) and (4·12) we obtain

$$z\sim 2\exp(-\tfrac{2}{3}\zeta^{\frac{3}{2}}-1),\qquad f(\zeta)\sim\frac{5}{16\zeta^2}. \tag{4·15}$$

Thus the preliminary conditions for the applicability of theorem B to equation (4·3) are satisfied. Following the notation of that theorem we define $\mathbf{D}'$ to be the remainder of the ζ-plane after the removal of the half-strips

$$|\mathscr{I}(\zeta\,\mathrm{e}^{\pm\frac{1}{3}\pi\mathrm{i}})|<\delta,\qquad \mathscr{R}(\zeta\,\mathrm{e}^{\pm\frac{1}{3}\pi\mathrm{i}})>(\tfrac{3}{2}\pi)^{\frac{2}{3}}-\delta$$

enclosing the cuts of $\mathbf{D}$, δ being a small arbitrary positive number. Taking a_1, a_2, a_3 to be the points at infinity on the rays $\arg\zeta=0$, $-\frac{2}{3}\pi$, $\frac{2}{3}\pi$ respectively, we find that the domains $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$ all coincide with $\mathbf{D}'$.

Applying theorem B we see that solutions $W_j(\zeta)$ $(j=1,2,3)$ of (4·3) exist such that as $n\to\infty$

$$W_j(\zeta)\sim P_j(n^{\frac{2}{3}}\zeta)\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{n^{2s}}+\frac{P_j'(n^{\frac{2}{3}}\zeta)}{n^{\frac{4}{3}}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{n^{2s}}, \tag{4·16}$$

uniformly with respect to ζ in $\mathbf{D}'$. The coefficients are defined by the equations $A_0(\zeta)=1$, and

$$\left.\begin{aligned} B_s(\zeta)&=\tfrac{1}{2}\zeta^{-\frac{1}{2}}\int_0^{\zeta}t^{-\frac{1}{2}}\{f(t)\,A_s(t)-A_s''(t)\}\,\mathrm{d}t,\\ A_{s+1}(\zeta)&=-\tfrac{1}{2}B_s'(\zeta)+\tfrac{1}{2}\int f(\zeta)\,B_s(\zeta)\,\mathrm{d}\zeta.\end{aligned}\right\} \tag{4·17}$$

We now express $J_n(nz)$ in terms of W_1, W_2 and W_3. From (4·2) it follows that

$$J_n(nz)=\left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{4}}W(\zeta), \tag{4·18}$$

where W denotes a linear combination of W_1, W_2 and W_3. Letting $\zeta\to+\infty$, $z\to 0$, we see immediately that if n is sufficiently large, W is a multiple of W_1.

Next, if we take $j=1$ and $\zeta=n^{-\frac{2}{3}}a_m'$ in (4·16), where a_m' is the mth negative real zero of $\mathrm{Ai}'(z)$, we obtain

$$W_1(\zeta)\sim\mathrm{Ai}(a_m')\left\{1+\sum_{s=1}^{\infty}\frac{A_s(n^{-\frac{2}{3}}a_m')}{n^{2s}}\right\}, \tag{4·19}$$

when n is large, uniformly with respect to m. For large m we have (see Appendix, equations (A 19) and (A 20))

$$a_m'=-\mu^{\frac{2}{3}}\{1+O(m^{-2})\},\qquad \mathrm{Ai}(a_m')=(-)^{m-1}\pi^{-\frac{1}{2}}\mu^{-\frac{1}{6}}\{1+O(m^{-2})\}, \tag{4·20}$$

where $\mu\equiv\frac{3}{8}\pi(4m-3)$. Substituting in the first of (4·14) we obtain

$$z(n^{-\frac{2}{3}}a_m')=\tfrac{2}{3}\mu n^{-1}+\tfrac{1}{2}\pi+O(m^{-1}), \tag{4·21}$$

and substituting this value for z in the well-known asymptotic formula (Watson 1944, p. 199)

$$J_n(nz)=(2/n\pi z)^{\frac{1}{2}}\{\cos(nz-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi)+O(z^{-1})\},$$

we have

$$J_n(nz)=(3/\mu\pi)^{\frac{1}{2}}\{\cos(\tfrac{2}{3}\mu-\tfrac{1}{4}\pi)+O(m^{-1})\}=(-)^{m-1}(3/\mu\pi)^{\frac{1}{2}}\{1+O(m^{-1})\}. \tag{4·22}$$

Combining the relations (4·18) to (4·22) and letting $m\to\infty$, we see that

$$J_n(nz)\div\left\{\left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{4}}W_1(\zeta)\right\}\sim\frac{\sqrt{2}}{n^{\frac{1}{3}}}\div\left\{1+\sum_{s=1}^{\infty}\frac{A_s(-\infty)}{n^{2s}}\right\}$$

for large n. Hence if we now prescribe

$$A_{s+1}(-\infty) = 0 \quad (s \geqslant 0), \tag{4·23}$$

we derive
$$J_n(nz) \sim \left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} \left\{\frac{\mathrm{Ai}\,(n^{\frac{2}{3}}\zeta)}{n^{\frac{1}{3}}} \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{n^{2s}} + \frac{\mathrm{Ai}'\,(n^{\frac{2}{3}}\zeta)}{n^{\frac{5}{3}}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{n^{2s}}\right\}, \tag{4·24}$$
as $n \to \infty$.

The result (4·24) is the desired expansion. The coefficients occurring in it are given by (4·17) with the arbitrary constants of integration fixed by (4·23). The expansion is uniform with respect to ζ in $\mathbf{D}'$, and hence it is uniform with respect to z in the sector $|\arg z| \leqslant \pi - \delta$, and if z lies in $\mathbf{K}$ it holds for *any* value of $\arg z$. When z lies outside these regions corresponding expansions may be deduced with the aid of the continuation formula

$$J_n(nz\,\mathrm{e}^{m\pi\mathrm{i}}) = \mathrm{e}^{mn\pi\mathrm{i}}\,J_n(nz),$$

m being any integer.

The significance of the eye-shaped domain $\mathbf{K}$ in relation to $J_n(nz)$ is now apparent. As $n \to \infty$, $|J_n(nz)|$ becomes exponentially small or exponentially large according as z lies inside† or outside $\mathbf{K}$, unless z happens to lie on the real axis outside $\mathbf{K}$ in which event $J_n(nz)$ oscillates boundedly.

The corresponding results for the Hankel functions $H_n^{(1)}(nz)$ and $H_n^{(2)}(nz)$ may be derived in a similar way. If $\zeta \to \infty\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}$ then $z \to \infty\,\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}$, and $H_n^{(1)}(nz)$ becomes exponentially small. Accordingly, $H_n^{(1)}(nz)$ must be a multiple of $W_2(\zeta)$ for all sufficiently large n. Similarly, $H_n^{(2)}(nz)$ is a multiple of $W_3(\zeta)$. The precise results may be verified to be

$$H_n^{(1)}(nz) \sim \frac{2\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}}{n^{\frac{1}{3}}} \left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} W_2(\zeta), \qquad H_n^{(2)}(nz) \sim \frac{2\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}}{n^{\frac{1}{3}}} \left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} W_3(\zeta), \tag{4·25}$$

where the coefficients $A_s(\zeta)$ and $B_s(\zeta)$ implied in the expansions for $W_2(\zeta)$ and $W_3(\zeta)$ are the same as in (4·24). In deriving these results use may be made of the fact that $\zeta^{\frac{1}{2}}B_s(\zeta) \to 0$ as $|\zeta| \to \infty$ in the sector $|\arg(-\zeta)| < \frac{2}{3}\pi$, which may itself be proved from (4·24) by setting $\zeta = n^{-\frac{2}{3}}a_m$, where a_m is the mth negative zero of $\mathrm{Ai}\,(z)$, and letting $m \to \infty$.

Combining the two results (4·25), we obtain

$$Y_n(nz) \sim -\left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} \left\{\frac{\mathrm{Bi}\,(n^{\frac{2}{3}}\zeta)}{n^{\frac{1}{3}}} \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{n^{2s}} + \frac{\mathrm{Bi}'\,(n^{\frac{2}{3}}\zeta)}{n^{\frac{5}{3}}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{n^{2s}}\right\}, \tag{4·26}$$

this expansion having the same region of validity as (4·24) and (4·25).

5. The expansion of $J_\nu(\nu z)$, $Y_\nu(\nu z)$ and the Hankel functions for large complex orders

As in § 3 let us write $\nu = n\,\mathrm{e}^{\mathrm{i}\vartheta}$. Then $(1-z^2)^{\frac{1}{4}}\,\zeta^{-\frac{1}{4}}\,J_\nu(\nu z)$ satisfies the equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2} = \{n^2\xi + \mathrm{e}^{-\frac{4}{3}\mathrm{i}\vartheta} f(\xi\,\mathrm{e}^{-\frac{2}{3}\mathrm{i}\vartheta})\}\,W \tag{5·1}$$

(cf. (4·3)), where $\xi \equiv \zeta\,\mathrm{e}^{\frac{2}{3}\mathrm{i}\vartheta}$ and $\zeta, f(\zeta)$ are defined by (4·11) and (4·12). The z-region comprising the sector $|\arg z| < \pi$ and the domain $\mathbf{K}$ of § 4 with any value of $\arg z$, is mapped on the ξ-plane cut along the rays

$$\arg \xi = \pm\tfrac{1}{3}\pi + \tfrac{2}{3}\vartheta, \qquad |\xi| \geqslant (\tfrac{3}{2}\pi)^{\frac{2}{3}}.$$

† It may be shown (Watson 1944, pp. 268–270) that if z lies inside $\mathbf{K}$ and n is *any* positive integer, then $|J_n(nz)| \leqslant 1$.

These cuts are indicated by the heavy lines in figure 7, the points E, E' having affixes $(\frac{3}{2}\pi)^{\frac{2}{3}}\exp(\mp\frac{1}{3}\mathrm{i}\pi+\frac{2}{3}\mathrm{i}\vartheta)$.

From theorem B it follows that solutions W_j $(j=1,2,3)$ of (5·1) exist such that

$$W_j \sim P_j(n^{\frac{2}{3}}\xi)\sum_{s=0}^{\infty}\frac{a_s(\xi)}{n^{2s}}+\frac{P_j'(n^{\frac{2}{3}}\xi)}{n^{\frac{4}{3}}}\sum_{s=0}^{\infty}\frac{b_s(\xi)}{n^{2s}}, \tag{5·2}$$

as $n\to\infty$. The regions of validity of these expansions do not, however, extend over the whole of the ξ-plane cut in the manner described. If, for example, $0<\vartheta<\frac{1}{2}\pi$ and the point a_1 of theorem B is taken to be at infinity on the positive real axis, then the definition of $\mathbf{D}_1$ shows that the expansion (5·2) with $j=1$ does not necessarily hold in the domain bounded by the branch of the level curve of $\exp(-\frac{2}{3}\xi^{\frac{3}{2}})$ through E and the cut through E. This excluded domain has been shaded in figure 7.

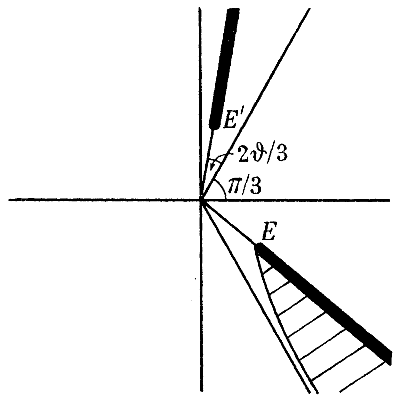

FIGURE 7. ξ-plane.

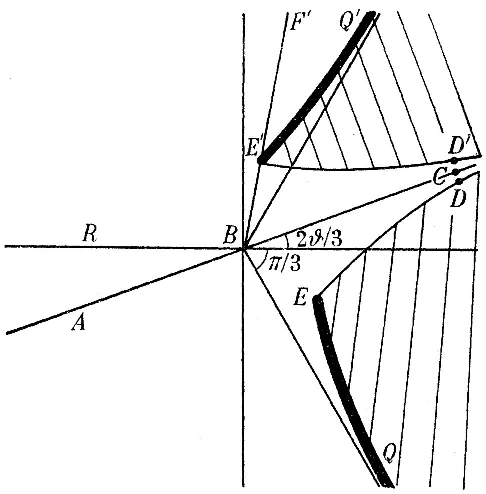

FIGURE 8. ξ-plane.

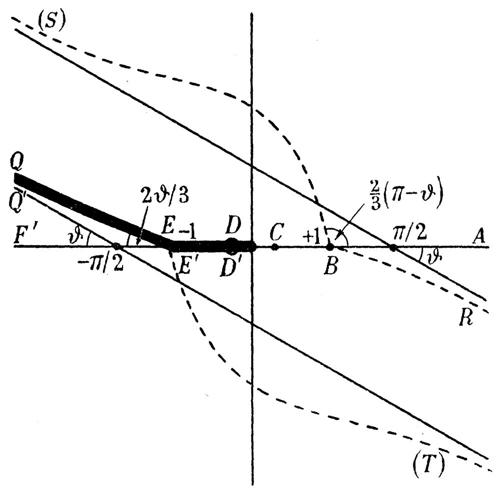

FIGURE 9. z-plane.

We are not, however, bound to make the cuts from E, E' along the rays indicated, and larger regions of validity result if we make them instead along the outward branches of the level curves of $\exp(-\frac{2}{3}\xi^{\frac{3}{2}})$ which pass through E, E'. From (4·7) it is seen that these curves are mapped in the ρ-plane as straight lines parallel to $\arg\rho=\pm\frac{1}{2}\pi-\vartheta$, and using this fact it is fairly easy to determine from (4·7) how they are mapped on the σ- and z-planes.

Figures 8 and 9 show corresponding points of the ξ, z transformation with a value of ϑ between 0 and $\frac{1}{2}\pi$. They should be compared with figures 3 and 6. In figure 8 the curves

EQ, $E'Q'$ are parts of level curves of $\exp(-\frac{2}{3}\xi^{\frac{3}{2}})$, and ED, $E'D'$ are the curves of figure 6 rotated round the origin through an angle $\frac{2}{3}\vartheta$. The unshaded part of the ξ-plane is mapped on the z-plane cut along the join of $z=0$ and $z=-1$ and the curve EQ whose parametric equation is defined by

$$z = \operatorname{sech}\sigma, \qquad \sigma - \tanh\sigma = -\mathrm{i}\pi + t\,\mathrm{e}^{\mathrm{i}(\frac{3}{2}\pi-\vartheta)} \quad (0 \leqslant t < \infty),$$

and

$$\sigma + \mathrm{i}\pi \sim (3t)^{\frac{1}{3}}\,\mathrm{e}^{\mathrm{i}(\frac{1}{2}\pi - \frac{1}{3}\vartheta)} \quad \text{as} \quad t \to 0.$$

This curve leaves $z=-1$ at an angle $\pi - \frac{2}{3}\vartheta$ with the real axis and has the line

$$\arg(z + \tfrac{1}{2}\pi) = \pi - \vartheta$$

as asymptote. In the cut z-plane, $\arg z$ takes its principal value everywhere except between $E'Q'$ and the negative real axis to the left of $z=-1$. In the theory of zeros (see §7) the negative real ξ-axis is of interest; it is mapped on the z-plane as the image (BR) in $z=0$ of the curve EQ.

As in §4 it is necessary to consider the mapping of the shaded zones of figure 7 on the z-plane. Without going into detail it may be stated that the upper zone is mapped on the interior of the continuous curve given by

$$z = \operatorname{sech}\sigma, \qquad \sigma - \tanh\sigma = \mathrm{i}\pi + t\,\mathrm{e}^{\mathrm{i}(\frac{1}{2}\pi-\vartheta)} \quad (0 \leqslant t < \infty),$$

and

$$\sigma \sim \mathrm{i}\pi + (3t)^{\frac{1}{3}}\,\mathrm{e}^{\mathrm{i}(\frac{1}{6}\pi - \frac{1}{3}\vartheta)} \quad \text{as} \quad t \to 0.$$

This curve starts from $z=-1$ at an angle $\frac{1}{3}\pi - \frac{2}{3}\vartheta$ with the real axis and winds itself round $z=0$ in the clockwise sense, tending to the origin as $t \to \infty$. The transformation is not one-to-one and $\arg z$ takes all possible values less than $-\pi$.

In the lower shaded zone the points

$$\xi = (\tfrac{3}{2}m\pi)^{\frac{2}{3}}\,\mathrm{e}^{\mathrm{i}(-\frac{1}{3}\pi + \frac{2}{3}\vartheta)} \quad (m = 1, 2, 3, \ldots),$$

are branch points of the function $z(\xi)$. If cuts are made along the outward branches of the level curves of $\exp(-\frac{2}{3}\xi^{\frac{3}{2}})$ through each of these points, then the whole zone is mapped on the interior z-region bounded by the curves EQ, BR, BS and $E'T$ of figure 9, $\arg z$ taking all possible values greater than π. The equation of BS is given by

$$z = \operatorname{sech}\sigma, \qquad \sigma - \tanh\sigma = t\,\mathrm{e}^{-\mathrm{i}(\frac{1}{2}\pi+\vartheta)} \quad (0 \leqslant t < \infty),$$

and

$$\sigma \sim (3t)^{\frac{1}{3}}\,\mathrm{e}^{-\mathrm{i}(\frac{1}{6}\pi + \frac{1}{3}\vartheta)} \quad \text{as} \quad t \to 0,$$

and $E'T$ is its image in $z=0$. BS leaves $z=1$ at an angle $\frac{2}{3}\pi - \frac{2}{3}\vartheta$ with the real axis and has the line $\arg(z - \frac{1}{2}\pi) = \pi - \vartheta$ as asymptote.

It has been assumed above that $0 < \vartheta < \frac{1}{2}\pi$. If $-\frac{1}{2}\pi < \vartheta < 0$, the roles of the upper and lower zones are interchanged.

Now that the correspondence between the z- and ξ-planes has been established we are in a position to express the Bessel functions in terms of the solutions W_j of (5·1). This is done in a manner similar to that of §4 for positive orders, and restoring afterwards the variables ν, ζ in place of n, ξ we find that *if the z-plane is cut in the manner indicated in figure* 9, $\arg z$ *having its principal value on the positive real axis, then the expansions* (4·24), (4·25) *and* (4·26) *hold with n replaced by* ν, *provided* $|\arg \nu| < \frac{1}{2}\pi$.

The region of validity of the expansions also extends to the z-region corresponding to the shaded ξ-zones of figure 8 which we have just examined. Asymptotic expansions valid for other ranges of $\arg z$ and $\arg \nu$ may be deduced with the aid of standard continuation formulae, given, for example, by Watson (1944, pp. 74–75).

6. Formulae for the coefficients and expansions for the derivatives

The coefficients $A_s(\zeta)$ and $B_s(\zeta)$ in the expansions for $J_\nu(\nu z)$, $Y_\nu(\nu z)$ and the Hankel functions are defined by (4·17) and (4·23), $f(\zeta)$ being given by (4·12). They are regular functions in the cut ζ-plane whether the cuts are made as in § 4 or as in § 5.

The first of (4·17) with $s=0$ may be integrated directly, using (4·12) and the first of (4·2), to give

$$B_0(\zeta) = -\frac{5}{48\zeta^2} - \frac{1}{\zeta^{\frac{1}{2}}}\left(\frac{v}{8} - \frac{5v^3}{24}\right) \tag{6·1}$$

where

$$v \equiv (1-z^2)^{-\frac{1}{2}}. \tag{6·2}$$

It is not practicable to continue this process, but explicit expressions for the higher coefficients may be obtained by the following procedure which is suggested in a remark of Cherry (1950, p. 250, footnote).

If ζ is taken temporarily to be positive and fixed then $n^{\frac{2}{3}}\zeta \to +\infty$ as $n\to\infty$, and $\mathrm{Ai}\,(n^{\frac{2}{3}}\zeta)$, $\mathrm{Ai}'\,(n^{\frac{2}{3}}\zeta)$ may be replaced in (4·24) by their asymptotic expansions (see Appendix, equations (A 1), (A2), (A3) and (A 6))

$$\mathrm{Ai}\,(n^{\frac{2}{3}}\zeta) \sim \frac{\exp\left(-\frac{2}{3}n\zeta^{\frac{3}{2}}\right)}{2n^{\frac{1}{6}}\zeta^{\frac{1}{4}}\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{(-)^s a_s}{n^s \zeta^{\frac{3}{2}s}}, \qquad \mathrm{Ai}'\,(n^{\frac{2}{3}}\zeta) \sim -\frac{n^{\frac{1}{6}}\zeta^{\frac{1}{4}}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}n\zeta^{\frac{3}{2}}\right) \sum_{s=0}^{\infty} \frac{(-)^s b_s}{n^s \zeta^{\frac{3}{2}s}}, \tag{6·3}$$

in which $a_0 = 1$, $b_0 = 1$, and

$$a_s = \frac{(2s+1)(2s+3)\dots(6s-1)}{s!\,(144)^s}, \qquad b_s = -\frac{6s+1}{6s-1} a_s. \tag{6·4}$$

The result so obtained must be equivalent to Debye's expansion (Watson 1944, p. 243), which can be written in the form

$$J_n(nz) \sim \frac{\exp\left(-\frac{2}{3}n\zeta^{\frac{3}{2}}\right)}{\sqrt{(2\pi n)}\,(1-z^2)^{\frac{1}{4}}} \sum_{s=0}^{\infty} \frac{\overline{U}_s}{n^s} \tag{6·5}$$

(cf. (2·13)), where $\overline{U}_s$ is given by the same formulae (2·19) and (2·20) as U_s but with u replaced by v, defined in (6·2). In consequence we derive the asymptotic equality

$$\sum_{s=0}^{\infty} \frac{\overline{U}_s}{n^s} \sim \sum_{s=0}^{\infty} \frac{(-)^s a_s}{n^s\zeta^{\frac{3}{2}s}} \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{n^{2s}} - \sum_{s=0}^{\infty} \frac{(-)^s b_s}{n^s\zeta^{\frac{3}{2}s}} \sum_{s=0}^{\infty} \frac{\zeta^{\frac{1}{2}}B_s(\zeta)}{n^{2s+1}}.$$

Equating coefficients of n^{-1}, we confirm the result (6·1), and with the aid of the relation

$$a_{2s}b_0 - a_{2s-1}b_1 + a_{2s-2}b_2 - \dots + a_0 b_{2s} = 0 \quad (s>0),$$

which may be derived from the Wronskian relation for Ai and Bi, we may show by induction that

$$A_s(\zeta) = \sum_{m=0}^{2s} b_m \zeta^{-\frac{3}{2}m}\,\overline{U}_{2s-m}, \qquad \zeta^{\frac{1}{2}}B_s(\zeta) = -\sum_{m=0}^{2s+1} a_m \zeta^{-\frac{3}{2}m}\overline{U}_{2s-m+1}. \tag{6·6}$$

Thus $A_s(\zeta)$ and $\zeta^{\frac{1}{2}}B_s(\zeta)$ are polynomials in $\zeta^{-\frac{3}{2}}$ and v. The validity of these results for all values of ζ in the cut plane follows by the principle of analytic continuation.

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Debye's formula may be regarded as a particular case of the present expansions. The same is true also of Meissel's expansions for Bessel functions of nearly equal argument and order (Watson 1944, pp. 245–248) and of the expansions given by the present writer (1952) for $J_\nu(z)$, $Y_\nu(z)$ and the Hankel functions when $z = \nu + \tau\nu^{\frac{1}{3}}$, τ being fixed and $|\nu|$ large. They may all be obtained from the expansions of §§ 4 and 5 by substituting appropriate expansions for the Airy functions.

The systematic tabulation of the coefficients $A_s(\zeta)$ and $B_s(\zeta)$ is most easily performed directly from the defining formulae (4·17), using processes of numerical differentiation and integration. For this purpose it is convenient to have the values of $A_1, A_2, \ldots$ at $\zeta = 0$, and they may be obtained by substituting $z = 1$ in (4·24) and comparing with Meissel's formula for $J_n(n)$ (Watson 1944, pp. 232–233). The first three values are found to be

$$A_1(0) = -\tfrac{1}{225} = -0{\cdot}00444\,44444\ldots, \qquad A_2(0) = +0{\cdot}00069\,3735\ldots,$$
$$A_3(0) = -0{\cdot}00035\,38\ldots.$$

Tables of $A_s(\zeta)$ and $B_s(\zeta)$ have been prepared for $s = 0, 1, 2, 3$ for the case of real positive arguments and orders. It transpires that these early coefficients decrease in magnitude with increasing s; in the range $-\infty < \zeta < \infty$ the maximum values of $|A_3|$ and $|B_3|$ are 0·00040 and 0·0010 approximately. Thus the series (4·24), (4·25) and (4·26) are particularly well suited to numerical applications.

Expansions for the derivatives

The expansion (4·24) may be differentiated term by term; if we write

$$\phi(\zeta) \equiv \left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} = \left(-\frac{2}{z}\frac{dz}{d\zeta}\right)^{\frac{1}{2}} \tag{6·7}$$

(cf. (4·2)), and replace n by ν, we obtain

$$\nu\frac{dz}{d\zeta}J'_\nu(\nu z) \sim \frac{\phi'(\zeta)}{\phi(\zeta)}J_\nu(\nu z) + \phi(\zeta)\left\{\frac{\mathrm{Ai}\,(\nu^{\frac{2}{3}}\zeta)}{\nu^{\frac{1}{3}}}\sum_{s=0}^{\infty}\frac{A'_s(\zeta)+\zeta B_s(\zeta)}{\nu^{2s}} + \frac{\mathrm{Ai}'\,(\nu^{\frac{2}{3}}\zeta)}{\nu^{\frac{5}{3}}}\sum_{s=-1}^{\infty}\frac{A_{s+1}(\zeta)+B'_s(\zeta)}{\nu^{2s}}\right\}.$$

Hence we derive the desired result

$$J'_\nu(\nu z) \sim -\psi(\zeta)\left\{\frac{\mathrm{Ai}\,(\nu^{\frac{2}{3}}\zeta)}{\nu^{\frac{4}{3}}}\sum_{s=0}^{\infty}\frac{C_s(\zeta)}{\nu^{2s}} + \frac{\mathrm{Ai}'\,(\nu^{\frac{2}{3}}\zeta)}{\nu^{\frac{2}{3}}}\sum_{s=0}^{\infty}\frac{D_s(\zeta)}{\nu^{2s}}\right\}, \tag{6·8}$$

where

$$\psi(\zeta) = 2/\{z\phi(\zeta)\}, \tag{6·9}$$

$$C_s(\zeta) = \chi(\zeta)\,A_s(\zeta) + A'_s(\zeta) + \zeta B_s(\zeta), \qquad D_s(\zeta) = A_s(\zeta) + \chi(\zeta)\,B_{s-1}(\zeta) + B'_{s-1}(\zeta), \tag{6·10}$$

and

$$\chi(\zeta) \equiv \frac{\phi'(\zeta)}{\phi(\zeta)} = \frac{1}{4\zeta} - \frac{z^2}{2}\left\{\frac{\zeta}{(1-z^2)^3}\right\}^{\frac{1}{2}} = \frac{4-z^2\{\phi(\zeta)\}^6}{16\zeta}. \tag{6·11}$$

The functions $\phi(\zeta)$, $\psi(\zeta)$, $\chi(\zeta)$, $C_s(\zeta)$ and $D_s(\zeta)$ are regular in the same regions as $z(\zeta)$, and (6·8) is valid under the same conditions as the expansion for $J_\nu(\nu z)$. The corresponding expansions for $Y'_\nu(\nu z)$ and the Hankel function derivatives are obtained by replacing Ai by the appropriate Airy functions.

Explicit formulae for $C_s(\zeta)$ and $D_s(\zeta)$ may be derived in a manner similar to that outlined above for $A_s(\zeta)$ and $B_s(\zeta)$; (6·8) is compared with Debye's expansion for $J'_n(nz)$. We find that

$$\zeta^{-\frac{1}{2}}C_s(\zeta) = -\sum_{m=0}^{2s+1} b_m \zeta^{-\frac{3}{2}m}\overline{V}_{2s-m+1}, \qquad D_s(\zeta) = \sum_{m=0}^{2s} a_m \zeta^{-\frac{3}{2}m}\overline{V}_{2s-m}, \tag{6·12}$$

where the functions $\overline{V}_s$ are given by (2·22) and (2·23) with u replaced by v, as defined by (6·2).

7. Zeros of $J_\nu(z)$ and $J'_\nu(z)$

Zeros of $J_n(z)$

We consider first the case when the order is real and positive. From (4·24) it follows that for large n the values of ζ corresponding to zeros of $J_n(nz)$ satisfy the equation (see also Olver 1954, equation (5·12))

$$\mathrm{Ai}\,(n^{\frac{2}{3}}\zeta)+(1+|\,n^{\frac{2}{3}}\zeta\,|^{\frac{1}{4}})^{-1}\exp\left(-\tfrac{2}{3}n\zeta^{\frac{3}{2}}\right)O(n^{-1})=0. \tag{7·1}$$

It is shown in the Appendix that the zeros of Ai (z) are all real and negative. If, in the usual notation, the sth zero is denoted by a_s, we deduce that the corresponding value of ζ is given by

$$\zeta=\alpha+\epsilon, \tag{7·2}$$

where

$$\alpha=n^{-\frac{2}{3}}a_s, \qquad \epsilon=O(n^{-\frac{8}{3}}), \tag{7·3}$$

and so $j_{n,s}$, the sth positive zero of $J_n(z)$, is given by

$$j_{n,s}=nz(\alpha+\epsilon)=nz(\alpha)+O(n^{-\frac{5}{3}}). \tag{7·4}$$

An asymptotic expansion for $j_{n,s}$ may be found as follows. From (4·24) and (7·2) we derive by expansion

$$W_1(\alpha)+\frac{\epsilon}{1!}W_1'(\alpha)+\frac{\epsilon^2}{2!}W_1''(\alpha)+\ldots=0, \tag{7·5}$$

where $W_1(\zeta)$ is defined in §4. Now from (4·16) with $j=1$ we obtain,† if m is any integer,

$$\left.\begin{aligned}W_1^{(2m)}(\zeta)&\sim n^{2m}\,\mathrm{Ai}\,(n^{\frac{2}{3}}\zeta)\sum_{r=0}^{\infty}\frac{A_r^{2m}(\zeta)}{n^{2r}}+n^{2m-\frac{4}{3}}\,\mathrm{Ai}'\,(n^{\frac{2}{3}}\zeta)\sum_{r=0}^{\infty}\frac{B_r^{2m}(\zeta)}{n^{2r}},\\ W_1^{(2m+1)}(\zeta)&\sim n^{2m}\,\mathrm{Ai}\,(n^{\frac{2}{3}}\zeta)\sum_{r=0}^{\infty}\frac{A_r^{2m+1}(\zeta)}{n^{2r}}+n^{2m+\frac{2}{3}}\,\mathrm{Ai}'\,(n^{\frac{2}{3}}\zeta)\sum_{r=0}^{\infty}\frac{B_r^{2m+1}(\zeta)}{n^{2r}},\end{aligned}\right\} \tag{7·6}$$

where

$$\left.\begin{aligned}A_r^{2m}&=\frac{\mathrm{d}}{\mathrm{d}\zeta}A_{r-1}^{2m-1}+\zeta B_r^{2m-1}, &\qquad A_r^{2m+1}&=\frac{\mathrm{d}}{\mathrm{d}\zeta}A_r^{2m}+\zeta B_r^{2m},\\ B_r^{2m}&=A_r^{2m-1}+\frac{\mathrm{d}}{\mathrm{d}\zeta}B_r^{2m-1}, &\qquad B_r^{2m+1}&=A_r^{2m}+\frac{\mathrm{d}}{\mathrm{d}\zeta}B_{r-1}^{2m}.\end{aligned}\right\} \tag{7·7}$$

Hence

$$W_1^{(2m)}(\alpha)\sim n^{2m-\frac{4}{3}}\,\mathrm{Ai}'\,(a_s)\sum_{r=0}^{\infty}\frac{B_r^{2m}(\alpha)}{n^{2r}},\qquad W_1^{(2m+1)}(\alpha)\sim n^{2m+\frac{2}{3}}\,\mathrm{Ai}'\,(a_s)\sum_{r=0}^{\infty}\frac{B_r^{2m+1}(\alpha)}{n^{2r}},$$

and substituting these values in (7·5), we obtain the asymptotic equality

$$\sum_{r=0}^{\infty}\frac{B_r(\alpha)}{n^{2r}}+\frac{\epsilon n^2}{1!}\sum_{r=0}^{\infty}\frac{B_r^1(\alpha)}{n^{2r}}+\frac{\epsilon^2n^2}{2!}\sum_{r=0}^{\infty}\frac{B_r^2(\alpha)}{n^{2r}}+\frac{\epsilon^3n^4}{3!}\sum_{r=0}^{\infty}\frac{B_r^3(\alpha)}{n^{2r}}+\ldots\sim 0. \tag{7·8}$$

From (7·7) we see that $B_0^1=1$, and so an expansion of ϵ exists of the form

$$\epsilon\sim\frac{\alpha_1}{n^2}+\frac{\alpha_2}{n^4}+\frac{\alpha_3}{n^6}+\ldots, \tag{7·9}$$

and substituting this relation in the first of (7·4) and expanding we obtain the desired result

$$j_{n,s}\sim n\sum_{r=0}^{\infty}\frac{p_r(\alpha)}{n^{2r}}, \tag{7·10}$$

† No question of the legitimacy of the repeated differentiation of the asymptotic series need arise; the results are otherwise obtainable from the differential equation (4·3) satisfied by $W_1(\zeta)$, and the recurrence relations (4·17) satisfied by $A_s(\zeta)$, $B_s(\zeta)$.

where α is given by (7·3) and the coefficients $p_r(\alpha)$ by the relation

$$\sum_{r=0}^{\infty} \frac{p_r(\alpha)}{n^{2r}} \sim z(\alpha) + \left(\frac{\alpha_1}{n^2} + \frac{\alpha_2}{n^4} + \dots\right) z'(\alpha) + \frac{1}{2!}\left(\frac{\alpha_1}{n^2} + \frac{\alpha_2}{n^4} + \dots\right)^2 z''(\alpha) + \dots. \tag{7·11}$$

The first three coefficients are readily found to be

$$p_0 = z, \qquad p_1 = \alpha_1 z', \qquad p_2 = \alpha_2 z' + \tfrac{1}{2}\alpha_1^2 z'', \tag{7·12}$$

where from (7·8) and (7·9),

$$\alpha_1 = -B_0, \qquad \alpha_2 = -(B_1 + \alpha_1 B_1^1 + \tfrac{1}{2}\alpha_1^2 B_0^2 + \tfrac{1}{6}\alpha_1^3 B_0^3). \tag{7·13}$$

If α is replaced by the general variable ζ, it is evident that for all values of r, $p_r(\zeta)$ is a polynomial in the functions $B_r^m(\zeta)$ and the derivatives $z^{(r)}(\zeta)$, and so is regular in the cut ζ-plane **D** of § 4. The expansion (7·10) is uniformly valid with respect to s, and the error on curtailing it at the term $p_r(\alpha)/n^{2r-1}$ is $O(n^{-2r-1})$. Thus the error term in (7·4) is really $O(n^{-1})$.

The value of $J_n'(z)$ at the zeros of $J_n(z)$ is another quantity of interest, and its asymptotic expansion may be conveniently obtained with the aid of the formula (Olver 1950, equation (3·4))

$$\pm J_n'(j_{n,s}) = \left(\tfrac{1}{2} j_{n,s} \frac{\mathrm{d}j_{n,s}}{\mathrm{d}s}\right)^{-\frac{1}{2}}, \tag{7·14}$$

in which s is regarded as a continuous variable. Using (7·3) and (7·10) we obtain

$$\frac{\mathrm{d}j_{n,s}}{\mathrm{d}s} \sim n^{\frac{1}{3}} \frac{\mathrm{d}a_s}{\mathrm{d}s} \sum_{r=0}^{\infty} \frac{p_r'(\alpha)}{n^{2r}}.$$

The result analogous to (7·14) for the Airy functions is given by (Olver 1950, equation (2·15))

$$\pm \mathrm{Ai}'(a_s) = \left(-\frac{\mathrm{d}a_s}{\mathrm{d}s}\right)^{-\frac{1}{2}}.$$

Also, remembering that $p_0 \equiv z$, we have from (6·7) and (6·9)

$$p_0 p_0' = -\tfrac{1}{2} z^2 \phi^2 = -2/\psi^2.$$

Combining these three results, and using (7·10) and (7·14), we derive

$$J_n'(j_{n,s}) \sim -\mathrm{Ai}'(a_s) \frac{\psi(\alpha)}{n^{\frac{2}{3}}} \left\{1 + \sum_{r=1}^{\infty} \frac{P_r(\alpha)}{n^{2r}}\right\}, \tag{7·15}$$

where the coefficients $P_r(\alpha)$ may be obtained from the relation

$$\left(1 + \sum_{r=1}^{\infty} \frac{P_r}{n^{2r}}\right)^{-2} \sim -\tfrac{1}{2}\psi^2 \left(\sum_{r=0}^{\infty} \frac{p_r}{n^{2r}}\right)\left(\sum_{r=0}^{\infty} \frac{p_r'}{n^{2r}}\right). \tag{7·16}$$

We find, for example, that

$$P_1 = \tfrac{1}{4}\psi^2(p_0 p_1' + p_1 p_0'), \qquad P_2 = \tfrac{3}{2}P_1^2 + \tfrac{1}{4}\psi^2(p_0 p_2' + p_1 p_1' + p_2 p_0'). \tag{7·17}$$

For numerical purposes the coefficients $p_r(\zeta)$, $\psi(\zeta)$ and $P_r(\zeta)$ of (7·10) and (7·15) may be pretabulated, and this has been done by the writer for the first few values of r. In the relevant range $0 < -\zeta < \infty$ the early coefficients *decrease* in magnitude; the values of $|p_1|$, $|p_2|$, $|p_3|$

do not exceed 0·015, 0·0012, 0·00045, and $|P_1|$, $|P_2|$, $|P_3|$ do not exceed 0·0070, 0·00036, 0·00006 4, respectively. Using the first four terms it is found, for example, that the series will give $j_{n,s}$ and $J'_n(j_{n,s})$ correct to at least ten significant figures when $n \geqslant 4$; the first term alone in each series gives four-figure accuracy at $n = 4$.

Zeros of $J'_n(z)$

When n is large the values of ζ corresponding to zeros of $J'_n(nz)$ are given by (cf. (6·8) and Olver 1954, equation (5·13))

$$\mathrm{Ai}'\,(n^{\frac{2}{3}}\zeta) + (1 + |\,n^{\frac{2}{3}}\zeta\,|^{\frac{1}{4}})\exp\left(-\tfrac{2}{3}n\zeta^{\frac{3}{2}}\right) O(n^{-\frac{2}{3}}) = 0.$$

The zeros a'_s of $\mathrm{Ai}'\,(z)$ are all real and negative (see Appendix), and the corresponding values of ζ are given by

$$\zeta = \beta + \eta,$$

where
$$\beta \equiv n^{-\frac{2}{3}} a'_s, \qquad \eta = O(n^{-\frac{4}{3}}). \tag{7·18}$$

Thus $j'_{n,s}$, the sth zero of $J'_n(z)$, is asymptotically given by

$$j'_{n,s} = nz(\beta + \eta) = nz(\beta) + O(n^{-\frac{1}{3}}).$$

Proceeding in a manner similar to that given above for $j_{n,s}$, we define

$$\left.\begin{aligned} C_r^{2m} &= \frac{\mathrm{d}}{\mathrm{d}\zeta} C_r^{2m-1} + \zeta D_r^{2m-1}, & C_r^{2m+1} &= \frac{\mathrm{d}}{\mathrm{d}\zeta} C_{r-1}^{2m} + \zeta D_r^{2m}, \\ D_r^{2m} &= C_r^{2m-1} + \frac{\mathrm{d}}{\mathrm{d}\zeta} D_{r-1}^{2m-1}, & D_r^{2m+1} &= C_r^{2m} + \frac{\mathrm{d}}{\mathrm{d}\zeta} D_r^{2m}. \end{aligned}\right\} \tag{7·19}$$

We find that
$$\eta \sim \frac{\beta_1}{n^2} + \frac{\beta_2}{n^4} + \frac{\beta_3}{n^6} + \ldots, \tag{7·20}$$

and
$$j'_{n,s} \sim n \sum_{r=0}^{\infty} \frac{q_r(\beta)}{n^{2r}}, \tag{7·21}$$

where the coefficients β_r and q_r are related by the formulae (7·8), (7·11) and (7·12) with the symbols B, p, ϵ and α, replaced by C, q, η and β, respectively. There is a difference between the two cases, however, because $C_0^1 = \zeta$, and in place of (7·13) we have

$$\beta_1 = -\zeta^{-1} C_0, \qquad \beta_2 = -\zeta^{-1}(C_1 + \beta_1 C_1^1 + \tfrac{1}{2}\beta_1^2 C_0^2 + \tfrac{1}{6}\beta_1^3 C_0^3). \tag{7·22}$$

In consequence, although $q_0(\zeta)$ $(\equiv z)$ is a regular function of ζ throughout $\mathbf{D}$, the higher coefficients $q_r(\zeta)$ have poles at $\zeta = 0$ of order $2r-1$ if $r \geqslant 1$. The error on curtailing the expansion (7·21) at the term $q_r(\beta)/n^{2r-1}$ is not $O(n^{-2r-1})$ but $O(n^{-\frac{2}{3}r-\frac{1}{3}})$, uniformly with respect to s.

The expansion for $J_n(j'_{n,s})$ corresponding to (7·15) may be derived in a similar way from the relations (Olver 1950, equations (3·9) and (2·15))

$$\pm J_n(j'_{n,s}) = \left(\frac{j'^2_{n,s} - n^2}{2j'_{n,s}} \frac{\mathrm{d}j'_{n,s}}{\mathrm{d}s}\right)^{-\frac{1}{2}}, \qquad \pm \mathrm{Ai}(a'_s) = \left(a'_s \frac{\mathrm{d}a'_s}{\mathrm{d}s}\right)^{-\frac{1}{2}}. \tag{7·23}$$

The result is
$$J_n(j'_{n,s}) \sim \mathrm{Ai}\,(a'_s) \frac{\phi(\beta)}{n^{\frac{1}{3}}} \left\{1 + \sum_{r=1}^{\infty} \frac{Q_r(\beta)}{n^{2r}}\right\}, \tag{7·24}$$

where ϕ is defined by (6·7) and the coefficients Q_r are calculable from the asymptotic equality

$$\left(1+\sum_{r=1}^{\infty}\frac{Q_r}{n^{2r}}\right)^{-2}\sim\frac{\phi^2}{2z\zeta}\left\{\left(\sum_{r=0}^{\infty}\frac{q_r}{n^{2r}}\right)^2-1\right\}\left(\sum_{r=0}^{\infty}\frac{q_r'}{n^{2r}}\right)\left(1+\sum_{r=1}^{\infty}\frac{q_r/q_0}{n^{2r}}\right)^{-1}. \tag{7·25}$$

The first two are given by

$$Q_1=\varpi_0 q_1'+\varpi_1 q_0', \qquad Q_2=\tfrac{3}{2}Q_1^2+\varpi_0 q_2'+\varpi_1 q_1'+\varpi_2 q_0', \tag{7·26}$$

where

$$\varpi_0=-\frac{\phi^2}{4\zeta}\left(q_0-\frac{1}{q_0}\right)=\frac{1}{z\phi^2}, \qquad \varpi_1=-\frac{\phi^2}{4\zeta}\left(q_1+\frac{q_1}{z^2}\right), \qquad \varpi_2=-\frac{\phi^2}{4\zeta}\left(q_2+\frac{q_2}{z^2}-\frac{q_1^2}{z^3}\right). \tag{7·27}$$

The functions $Q_r(\zeta)$ have poles of order not exceeding† $2r$ at $\zeta=0$, and if the series inside the braces of (7·24) is curtailed at the term $Q_r(\beta)/n^{2r}$, the error in the value for $J_n(j'_{n,s})$ is uniformly $O(n^{-\frac{4}{3}r-1})$.

The fact that the forms of the error terms of (7·21) and (7·24) are weaker than those of (7·10) and (7·15) is somewhat surprising. It arises essentially from the process of reversion used, which is not so effective applied to (6·8) as it is to (4·24). However, in numerical applications the difference is only marked for very low values of s. In these exceptional cases the 'lost' accuracy may always be recovered by solving the equation $J_n'(nz)=0$ by successive approximation, evaluating the trial values of $J_n'(nz)$ by means of (6·8).

Zeros of functions of complex order

The extension of the results of this section to the half-plane $|\arg\nu|<\frac{1}{2}\pi$ presents no difficulty. The expansions (7·10), (7·15), (7·21) and (7·24) remain valid with n replaced by the complex variable ν and

$$\alpha=\nu^{-\frac{2}{3}}a_s, \qquad \beta=\nu^{-\frac{2}{3}}a_s',$$

these quantities now being complex. From § 5 it is seen that the points $z(\alpha)$ and $z(\beta)$ lie on the curve BR of figure 9, and so when $|\arg\nu|<\frac{1}{2}\pi$ and $|\nu|$ is large the zeros of $J_\nu(\nu z)$ and $J_\nu'(\nu z)$ lie asymptotically close to this curve.

8. Zeros of $Y_n(z)$ and $Y_n'(z)$

The analysis of § 7 may be repeated with equation (4·26) in place of (4·24). If, in the usual notation, $y_{n,s}$ and $y'_{n,s}$ denote the sth positive zeros of $Y_n(z)$ and $Y_n'(z)$ respectively, then the expansions (7·10), (7·15), (7·21) and (7·24) evidently remain valid with the symbols j and J replaced by y and Y respectively, provided that Ai and a are replaced by $-$Bi and b; the quantities α, β are now given by

$$\alpha=n^{-\frac{2}{3}}b_s, \qquad \beta=n^{-\frac{2}{3}}b_s',$$

(cf. (7·3) and (7·18)), where b_s, b_s' are the sth negative zeros of Bi(z), Bi$'(z)$ respectively.

An important difference emerges, however, between the cases of Y_n and J_n. Whereas the zeros of J_n are all real and positive, Y_n has in addition to its real zeros a number which are complex. In this section we investigate the distribution of the complex zeros.

It is shown in the Appendix that the function Bi(z) has a string of zeros $\beta_1, \beta_2, \ldots$, lying in the sector $\frac{1}{3}\pi<\arg z<\frac{1}{2}\pi$, together with a conjugate set $\bar\beta_1, \bar\beta_2, \ldots$, in the conjugate sector.

† It has been verified that the order is actually $2r-1$ for $r=1$, 2, 3, and for these cases the truncation error is $O(n^{-\frac{4}{3}r-\frac{2}{3}})$.

If they are arranged in ascending order of modulus magnitude then the asymptotic expansion of β_s for large s is given by (cf. Appendix equations (A19) and (A22))

$$\beta_s \sim e^{\frac{1}{3}\pi i}\,\lambda^{\frac{2}{3}}\left(1+\tfrac{5}{48}\lambda^{-2}-\tfrac{5}{36}\lambda^{-4}+\tfrac{77125}{82944}\lambda^{-6}-\dots\right),$$

where

$$\lambda = \tfrac{3}{8}\{(4s-1)\,\pi+2i\ln 2\}.$$

Thus the large zeros in this sector lie asymptotically close to $\arg z = \frac{1}{3}\pi$, but even the small zeros all lie very near to this ray; numerical calculation (see § 10) shows, for example, that $\arg \beta_1 = \frac{1}{3}\pi + 0{\cdot}095\,\dots$.

Corresponding to β_s there is, for large n, a zero $\overline{\eta}_{n,s}$ of $Y_n(z)$ whose asymptotic expansion is given by

$$\overline{\eta}_{n,s} \sim \sum_{r=0}^{\infty} \frac{p_r(n^{-\frac{2}{3}}\beta_s)}{n^{2r-1}} \tag{8·1}$$

(cf. (7·10)), where the functions $p_r(\zeta)$ are those of § 7. This expansion is not, however, uniformly valid with respect to all s; if $s \geqslant n$ the point $\zeta = n^{-\frac{2}{3}}\beta_s$ lies asymptotically close to the cut along the ray $\arg\zeta = \frac{1}{3}\pi$ from $\zeta = (\frac{3}{2}\pi)^{\frac{2}{3}}\,e^{\frac{1}{3}\pi i}$ to infinity. This cut is a boundary of the region of validity of (4·26) and the expansion (8·1) is uniformly valid only when $s \leqslant \kappa n$, where κ is any fixed number in the range $0<\kappa<1$.

The approximate distribution of $\overline{\eta}_{n,s}$ may be found by curtailing (8·1) at its first term, given by

$$\overline{\eta}_{n,s} = nz(n^{-\frac{2}{3}}\beta_s)+O(n^{-1}) \tag{8·2}$$

(see (7·12)), where $z(\zeta)$ is the function defined in § 4. If $s<n$, the point $\zeta = n^{-\frac{2}{3}}\beta_s$ lies close to the segment $BP'E'$ (figure 6) of the ray $\arg\zeta = \frac{1}{3}\pi$; this is mapped in the z-plane on the curve $BP'E'$ (figure 3), the lower boundary of the domain **K**. In consequence, the zeros $\overline{\eta}_{n,s}$ lie close to, and in fact just outside, the lower boundary of the domain† **nK**.

The relations (8·1) and (8·2) naturally break down when the value of s approaches n, that is, for zeros lying in the neighbourhood of the transition point $z=-1$. In order to achieve a more complete account we must consider the expansions of $Y_n(nz)$ corresponding to (4·26) for general phase ranges of z. Before proceeding to this we record some properties of a relevant Airy function.

The Airy function $\mathrm{Di}_m(z)$

This is defined by the equation

$$\mathrm{Di}_m(z) = mi\,\mathrm{Ai}(z)-\mathrm{Bi}(z), \tag{8·3}$$

where m is a constant, which we restrict here to be real. Other Airy functions may be expressed in terms of $\mathrm{Di}_m(z)$. Using the equation (*British Association Mathematical Tables* 1946, p. B 9)

$$\mathrm{Ai}(e^{\frac{2}{3}\pi i}z) = \tfrac{1}{2}e^{\frac{1}{3}\pi i}\{\mathrm{Ai}(z)-i\,\mathrm{Bi}(z)\},$$

we readily verify that

$$\mathrm{Di}_0(z) = -\mathrm{Bi}(z), \qquad \mathrm{Di}_1(z) = 2e^{\frac{5}{6}\pi i}\,\mathrm{Ai}(e^{-\frac{2}{3}\pi i}z), \qquad \mathrm{Di}_3(z) = 2e^{\frac{2}{3}\pi i}\,\mathrm{Bi}(e^{\frac{2}{3}\pi i}z). \tag{8·4}$$

Relations connecting functions having different values of m are

$$\mathrm{Di}_{-m}(z) = \overline{\mathrm{Di}_m(\overline{z})}, \qquad \mathrm{Di}_m(e^{\frac{2}{3}\pi i}z) = \tfrac{1}{2}(m-1)\,e^{-\frac{2}{3}\pi i}\,\mathrm{Di}_{(3+m)/(1-m)}(z). \tag{8·5}$$

† The 'domain **nK**' means the domain **K** magnified by the factor n.

The asymptotic expansions of $\mathrm{Di}_m(z)$ and $\mathrm{Di}'_m(z)$ for large $|z|$ may be obtained without difficulty from those of $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ given by equations (A 1) to (A 10) in the Appendix. Using the same notation we find that

$$\left.\begin{aligned}\mathrm{Di}_m(-z) &\sim \sqrt{\left(\frac{m^2-1}{\pi}\right)}\,\mathrm{i}z^{-\frac{1}{4}}\left\{\cos\left(\xi-\tfrac{1}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{m+1}{m-1}\right)P(\xi)+\sin\left(\xi-\tfrac{1}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{m+1}{m-1}\right)Q(\xi)\right\} \quad (m^2>1),\\ \mathrm{Di}'_m(-z) &\sim \sqrt{\left(\frac{m^2-1}{\pi}\right)}\,\mathrm{i}z^{\frac{1}{4}}\left\{\cos\left(\xi-\tfrac{3}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{m+1}{m-1}\right)R(\xi)+\sin\left(\xi-\tfrac{3}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{m+1}{m-1}\right)S(\xi)\right\} \quad (m^2>1),\end{aligned}\right\}$$

$$\begin{aligned}\mathrm{Di}_m(z\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}) &\sim \sqrt{\left(\frac{2(m-1)}{\pi}\right)}\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}\,z^{-\frac{1}{4}}\left\{\cos\left(\xi+\tfrac{1}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{m-1}{2}\right)P(\xi)+\sin\left(\xi+\tfrac{1}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{m-1}{2}\right)Q(\xi)\right\} \quad (m>\\ \mathrm{Di}'_m(z\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}) &\sim \sqrt{\left(\frac{2(m-1)}{\pi}\right)}\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,z^{\frac{1}{4}}\left\{\cos\left(\xi-\tfrac{1}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{m-1}{2}\right)R(\xi)+\sin\left(\xi-\tfrac{1}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{m-1}{2}\right)S(\xi)\right\} \quad (m>\end{aligned}$$

$$\begin{aligned}\mathrm{Di}_m(z\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}) &\sim \sqrt{\left(\frac{2(1-m)}{\pi}\right)}\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\,z^{-\frac{1}{4}}\left\{\cos\left(\xi+\tfrac{3}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{1-m}{2}\right)P(\xi)+\sin\left(\xi+\tfrac{3}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{1-m}{2}\right)Q(\xi)\right\} \quad (m<\\ \mathrm{Di}'_m(z\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}) &\sim \sqrt{\left(\frac{2(1-m)}{\pi}\right)}\,\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}\,z^{\frac{1}{4}}\left\{\cos\left(\xi-\tfrac{3}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{1-m}{2}\right)R(\xi)+\sin\left(\xi-\tfrac{3}{4}\pi+\frac{\mathrm{i}}{2}\ln\frac{1-m}{2}\right)S(\xi)\right\} \quad (m<1\end{aligned}$$

each of these expansions being valid in the sector $|\arg z|<\frac{2}{3}\pi$. Expansions for $\mathrm{Di}_m(z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})$ and $\mathrm{Di}'_m(z\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})$ corresponding to (8·7) and (8·8) may be obtained by changing the signs of i and m in the right-hand sides of these relations (see the first of (8·5)).

From these results it is seen that $\mathrm{Di}_m(z)$ and $\mathrm{Di}'_m(z)$ have strings of zeros lying asymptotically close to the rays $\arg z=\pi$, $\frac{1}{3}\pi$ and $-\frac{1}{3}\pi$. We denote those of $\mathrm{Di}_m(z)$ by $\mathrm{d}_{m,s}$, $\delta_{m,s}$ and $\epsilon_{m,s}$ $(s=1,2,\ldots)$, respectively, and those of $\mathrm{Di}'_m(z)$ by $d'_{m,s}$, $\delta'_{m,s}$ and $\epsilon'_{m,s}$ $(s=1,2,\ldots)$, respectively, and suppose each string to be arranged in ascending order of modulus.

The asymptotic expansions of these zeros when s is large may be found by reversion of (8·6) to (8·8). Using the notation $T(\lambda)$, $U(\mu)$, $V(\lambda)$ and $W(\mu)$ defined by equation (A 19) of the Appendix, we find for the group of zeros near $\arg z=\pi$

$$\left.\begin{aligned}&d_{m,s}\sim -T(\lambda), && d'_{m,s}\sim -U(\mu),\\ &\mathrm{Di}'_m(d_{m,s})\sim(-)^{s-1}(m^2-1)^{\frac{1}{2}}\,\mathrm{i}V(\lambda), && \mathrm{Di}_m(d'_{m,s})\sim(-)^{s-1}(m^2-1)^{\frac{1}{2}}\,\mathrm{i}W(\mu),\\ &\text{where}\quad \lambda=\frac{3}{8}\left\{(4s-1)\,\pi-2\mathrm{i}\ln\frac{m+1}{m-1}\right\}, && \mu=\frac{3}{8}\left\{(4s-3)\,\pi-2\mathrm{i}\ln\frac{m+1}{m-1}\right\}\quad(m^2>1).\end{aligned}\right\}\quad(8\cdot9)$$

For the zeros near $\arg z=\frac{1}{3}\pi$, we find

$$\left.\begin{aligned}&\delta_{m,s}\sim \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,T(\lambda), && \delta'_{m,s}\sim \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,U(\mu),\\ &\mathrm{Di}'_m(\delta_{m,s})\sim(-)^s\{2(m-1)\}^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,V(\lambda), && \mathrm{Di}_m(\delta'_{m,s})\sim(-)^s\{2(m-1)\}^{\frac{1}{2}}\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}\,W(\mu)\quad(m>1),\\ &\mathrm{Di}'_m(\delta_{m,s})\sim(-)^s\{2(1-m)\}^{\frac{1}{2}}\,\mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}\,V(\lambda), && \mathrm{Di}_m(\delta'_{m,s})\sim(-)^s\{2(1-m)\}^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\,W(\mu)\quad(m<1),\\ &\text{where}\quad \lambda=\frac{3}{8}\left\{(4s-3)\,\pi-2\mathrm{i}\ln\frac{m-1}{2}\right\}, && \mu=\frac{3}{8}\left\{(4s-1)\,\pi-2\mathrm{i}\ln\frac{m-1}{2}\right\}\quad(m>1),\\ &\qquad\quad\ \lambda=\frac{3}{8}\left\{(4s-1)\,\pi-2\mathrm{i}\ln\frac{1-m}{2}\right\}, && \mu=\frac{3}{8}\left\{(4s-3)\,\pi-2\mathrm{i}\ln\frac{1-m}{2}\right\}\quad(m<1).\end{aligned}\right\}\quad(8\cdot10$$

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Finally, for the zeros near $\arg z = -\frac{1}{3}\pi$, we find

$$\left.\begin{array}{ll} \epsilon_{m,s} \sim e^{-\frac{1}{3}\pi i}\, T(\lambda), & \epsilon'_{m,s} \sim e^{-\frac{1}{3}\pi i}\, U(\mu), \\ \mathrm{Di}'_m(\epsilon_{m,s}) \sim (-)^s \{2(1+m)\}^{\frac{1}{2}} e^{-\frac{2}{3}\pi i}\, V(\lambda), & \mathrm{Di}_m(\epsilon'_{m,s}) \sim (-)^s \{2(1+m)\}^{\frac{1}{2}} e^{-\frac{1}{3}\pi i}\, W(\mu) \quad (m>-1), \\ \mathrm{Di}'_m(\epsilon_{m,s}) \sim (-)^s \{2(-1-m)\}^{\frac{1}{2}} e^{-\frac{1}{3}\pi i}\, V(\lambda), & \mathrm{Di}_m(\epsilon'_{m,s}) \sim (-)^s \{2(-1-m)\}^{\frac{1}{2}} e^{-\frac{2}{3}\pi i}\, W(\mu) \quad (m<-1), \\ \text{where} \quad \lambda = \frac{3}{8}\left\{(4s-1)\,\pi + 2i \ln \frac{1+m}{2}\right\}, & \mu = \frac{3}{8}\left\{(4s-3)\,\pi + 2i\ln\frac{1+m}{2}\right\} \quad (m>-1), \\ \lambda = \frac{3}{8}\left\{(4s-3)\,\pi + 2i\ln\frac{(-1-m)}{2}\right\}, & \mu = \frac{3}{8}\left\{(4s-1)\,\pi + 2i\ln\frac{(-1-m)}{2}\right\} \quad (m<-1). \end{array}\right\} \tag{8·11}$$

It can be verified by contour integration that $\mathrm{Di}_m(z)$ and $\mathrm{Di}'_m(z)$ have no zeros other than those enumerated above.

Zeros of $Y_n(z)$ and $Y'_n(z)$ for general phase ranges of z, when n is an integer

If m is any integer we have (Watson 1944, p. 75),

$$Y_n(z\, e^{m\pi i}) = e^{-mn\pi i}\, Y_n(z) + 2i \sin mn\pi \cot n\pi J_n(z). \tag{8·12}$$

Hence, if n is an integer,

$$Y_n(z\, e^{m\pi i}) = (-)^{mn} \{Y_n(z) + 2imJ_n(z)\}. \tag{8·13}$$

Combining this result with (4·24) and (4·26), we see that

$$Y_n(nz\, e^{m\pi i}) \sim (-)^{mn} \left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}} \left\{\frac{\mathrm{Di}_{2m}(n^{\frac{2}{3}}\zeta)}{n^{\frac{1}{3}}} \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{n^{2s}} + \frac{\mathrm{Di}'_{2m}(n^{\frac{2}{3}}\zeta)}{n^{\frac{5}{3}}} \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{n^{2s}}\right\}, \tag{8·14}$$

for large n, where ζ, $A_s(\zeta)$ and $B_s(\zeta)$ are the same functions as in § 4, and $\mathrm{Di}_m(z)$ is defined by (8·3). The expansion (8·14) is valid when $|\arg z| < \pi$, but we shall use it only in the half-plane $|\arg z| \leqslant \frac{1}{2}\pi$.

Corresponding to a zero δ of $\mathrm{Di}_{2m}(z)$, there is a zero $z = \eta$ of $Y_n(z\, e^{m\pi i})$ given by (cf. (8·2))

$$\eta = nz(n^{-\frac{2}{3}}\delta) + O(n^{-1}). \tag{8·15}$$

Consider first the zeros corresponding to $d_{2m,s}$ $(s = 1, 2, \ldots)$. If s is large, and $m \neq 0$, we have from (8·9),

$$n^{-\frac{2}{3}} d_{2m,s} = -\left(\frac{3}{2n}\right)^{\frac{2}{3}} \left\{(s-\tfrac{1}{4})\,\pi - \frac{i}{2}\ln\frac{2m+1}{2m-1}\right\}^{\frac{2}{3}} \{1 + O(s^{-2})\}.$$

If $s \gg n$, so that $n^{-\frac{2}{3}} d_{2m,s}$ is large, we deduce from this result, (8·15) and the first of (4·14) that

$$\eta = (s-\tfrac{1}{4})\,\pi + \tfrac{1}{2}n\pi - \frac{i}{2}\ln\frac{2m+1}{2m-1} + O(n^2/s) + O(n^{-1}).$$

Letting $s \to \infty$, we see that the curve of zeros has the asymptote

$$\mathscr{I} z = -\tfrac{1}{2}\ln\frac{2m+1}{2m-1} + O(n^{-1}) \quad (m \neq 0). \tag{8·16}$$

This result may be otherwise obtained from (8·13) and Hankel's expansions (Watson 1944, p. 199) for $J_n(z)$ and $Y_n(z)$. In this way we find that if n is a positive integer or zero then the asymptote is given by

$$\mathscr{I} z = -\tfrac{1}{2}\ln\left|\frac{2m+1}{2m-1}\right|. \tag{8·17}$$

Thus the error term $O(n^{-1})$ in (8·16) is in fact zero.

Consider next the zeros corresponding to $\delta_{2m,s}$ $(s = 1, 2, \ldots)$. The points $\zeta = n^{-\frac{2}{3}}\delta_{2m,s}$, like the points $\zeta = n^{-\frac{2}{3}}\beta_s$ in (8·1), all lie near to the ray $\arg\zeta = \frac{1}{3}\pi$, and hence the corresponding zeros of $Y_n(nz\,\mathrm{e}^{m\pi\mathrm{i}})$ lie near to the lower boundary $BP'E'$ (figure 3) of the domain **K**. These zeros may be regarded as lying on a continuous curve, and it is of interest to know where this curve meets the imaginary z-axis, and how many zeros lie to the right of this axis.

If $m \geqslant 1$, we have from (8·10),

$$n^{-\frac{2}{3}}\delta_{2m,s} = \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\left(\frac{3}{2n}\right)^{\frac{2}{3}}\{(s-\tfrac{3}{4})\,\pi - \tfrac{1}{2}\mathrm{i}\ln|m-\tfrac{1}{2}|\}^{\frac{2}{3}}\{1+O(s^{-2})\}. \tag{8·18}$$

The curve in the ζ-plane passing through the zeros is obtained by allowing s to be a continuous real variable. Now from § 4, the points

$$\zeta = \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,(\tfrac{3}{4}\pi)^{\frac{2}{3}} \quad \text{and} \quad z = \bar{z}_0 \equiv -\mathrm{i}(t_0^2-1)^{\frac{1}{2}} = -\mathrm{i}\,0{\cdot}66274\ldots, \tag{8·19}$$

where t_0 is the positive root of $\coth t = t$, are corresponding points. At $z = \bar{z}_0$ we have, from (4·2),

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)_{z=\bar{z}_0} = -z\left(\frac{\zeta}{1-z^2}\right)^{\frac{1}{2}} = \mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}\,(\tfrac{3}{4}\pi)^{\frac{1}{3}}\,(1-t_0^{-2})^{\frac{1}{2}}.$$

Setting $s = \frac{1}{2}n + \frac{3}{4}$ in (8·18), we find that the imaginary z-axis and the curve of zeros of $Y_n(nz\,\mathrm{e}^{m\pi\mathrm{i}})$ intersect at

$$z = \bar{z}_0 + \frac{\mathrm{i}}{2n}(1-t_0^{-2})^{\frac{1}{2}}\ln|m-\tfrac{1}{2}| + O(n^{-2}). \tag{8·20}$$

The zeros lying to the right of the imaginary z-axis are seen from (8·18) to be those for which $s - \frac{3}{4} < \frac{1}{2}n$; there are thus $[\frac{1}{2}n + \frac{1}{2}]$ such zeros.

When $m \leqslant 0$ the equation corresponding to (8·18) is obtained by replacing $\frac{3}{4}$ by $\frac{1}{4}$; the equation (8·20) remains valid, but the number of zeros to the right of the imaginary axis is now $[\frac{1}{2}n]$.

The analogous results concerning the zeros corresponding to $\epsilon_{2m,s}$ may be deduced by the use of the conjugate property (see the first of (8·5)),

$$\epsilon_{m,s} = \bar{\delta}_{-m,s}.$$

The curve of these zeros lies near the upper boundary of **K** and intersects the imaginary axis at the point

$$z = z_0 - \frac{\mathrm{i}}{2n}(1-t_0^{-2})^{\frac{1}{2}}\ln|m+\tfrac{1}{2}| + O(n^{-2}). \tag{8·21}$$

The number of zeros to the right of this axis is $[\frac{1}{2}n]$ or $[\frac{1}{2}n+\frac{1}{2}]$ according as $m \geqslant 0$ or $m \leqslant -1$.

We are now in a position to assess the asymptotic distribution of zeros of $Y_n(z\,\mathrm{e}^{m\pi\mathrm{i}})$ in the wider range $|\arg z| \leqslant \pi$. In the quadrant $\frac{1}{2}\pi \leqslant \arg z \leqslant \pi$ the zeros are the same as those of $Y_n(z\,\mathrm{e}^{(m+1)\pi\mathrm{i}})$ rotated through an angle π. There is an infinite string of zeros here if $m \geqslant -1$, having the asymptote

$$\mathscr{I}z = \tfrac{1}{2}\ln\left|\frac{2m+3}{2m+1}\right|$$

(cf. (8·17)), and the curve of zeros near the upper boundary of **nK** continues from the point given by (8·21) towards $z = -n$; the number of zeros on this segment is $[\frac{1}{2}n+\frac{1}{2}]$ or $[\frac{1}{2}n]$ according as $m \geqslant 0$ or $m \leqslant -1$. Combining this with the result for $|\arg z| \leqslant \frac{1}{2}\pi$, we see that for all values of m the total number of zeros on the upper curve is n.

Similarly, there is an infinite string of zeros in the quadrant $-\pi \leqslant \arg z \leqslant -\frac{1}{2}\pi$ if $m \leqslant 1$, and for any m the lower curve from $z = n$ to $z = -n$ also contains n zeros.

These results are illustrated in figures 10, 11 and 12. Figures 10 and 12 are conjugate to each other. The continuous curves are the boundaries of $\mathbf{nK}$; they intersect the imaginary axis at the points nz_0 and $n\bar{z}_0$, where $\bar{z}_0$ is given by (8·19). The broken lines indicate the

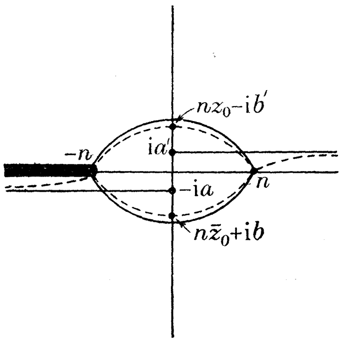

FIGURE 10. $m \leqslant -1$.

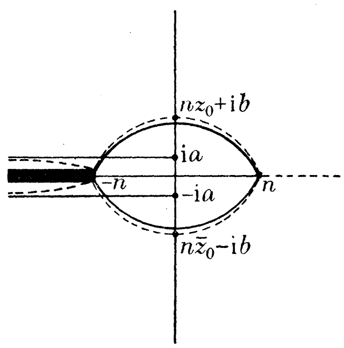

FIGURE 11. $m = 0$.

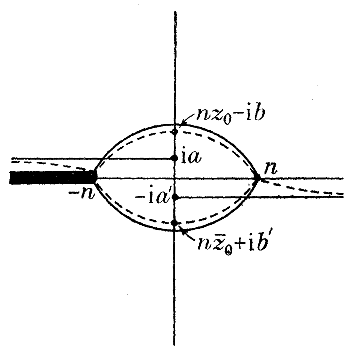

FIGURE 12. $m \geqslant 1$.

Distribution of the zeros ---- of $Y_n(z)$ in $(2m-1)\,\pi \leqslant \arg z \leqslant (2m+1)\,\pi$, when n is an integer.

curves on which the zeros lie. The displacements of the asymptotes of the infinite branches from the real axis are a and a', and the distances between the broken and continuous curves where they cross the imaginary axis are b and b', where, when $m \neq 0$, a, a', b and b' are given by

$$\left.\begin{aligned} a &= \tfrac{1}{2}\ln\frac{4\,|\,m\,|+3}{4\,|\,m\,|+1}, \qquad & b &= \tfrac{1}{2}(1-t_0^{-2})^{\frac{1}{2}}\ln\left(2\,|\,m\,|+\tfrac{1}{2}\right)+O(n^{-1}),\\ a' &= \tfrac{1}{2}\ln\frac{4\,|\,m\,|+1}{4\,|\,m\,|-1}, \qquad & b' &= \tfrac{1}{2}(1-t_0^{-2})^{\frac{1}{2}}\ln\left(2\,|\,m\,|-\tfrac{1}{2}\right)+O(n^{-1}).\end{aligned}\right\} \tag{8·22}$$

For $m = 0$ we have

$$a = \tfrac{1}{2}\ln 3 = 0{\cdot}54931\ldots, \qquad b = \tfrac{1}{2}(1-t_0^{-2})^{\frac{1}{2}}\ln 2+O(n^{-1}) = 0{\cdot}19146\ldots+O(n^{-1}). \tag{8·23}$$

There are n zeros on each of the finite curves.

The zeros of $Y_n'(z)$ may be investigated in a similar manner, and the results are very similar. Figures 10, 11 and 12 are also applicable to this function, and the number of zeros on each of the finite curves is again n. The significant change is in the form of the error term (cf. (7·22)); in the formulae corresponding to (8·2), (8·15), (8·22) and (8·23) the error term is $O(n^{-\frac{1}{3}})$ and not $O(n^{-1})$.

The results obtained in this section are valid for large values of n. We can, however, determine by contour integration the number of zeros of $Y_n(z\,\mathrm{e}^{m\pi\mathrm{i}})$ and $Y_n'(z\,\mathrm{e}^{m\pi\mathrm{i}})$ in the strip $0 \leqslant \mathscr{R}z \leqslant (s+\frac{1}{2}n)\pi$ when n is any positive integer or zero, and s is an integer. We find, without difficulty, that if s is sufficiently large $Y_n(z\,\mathrm{e}^{m\pi\mathrm{i}})$ and $Y_n'(z\,\mathrm{e}^{m\pi\mathrm{i}})$ each have $n+s$ zeros in this strip, unless $m = 0$ and n is odd, in which event $Y_n(z)$ has $n+s-1$ zeros and $Y_n'(z)$ has $n+s+1$ zeros. Thus when n is a positive integer $Y_n(z)$ *and* $Y_n'(z)$ *have no zeros other than those indicated by the asymptotic formula* (8·15).

Tables for the numerical evaluation of the first term in the various formulae for the zeros and associated values are given in § 10.

Zeros of $Y_n(z)$ and $Y_n'(z)$ when n is half an odd integer

If we substitute $n = n_0 + \epsilon$ in (8·12), where n_0 is an integer and $0 \leqslant \epsilon < 1$, we obtain

$$Y_n(z\,\mathrm{e}^{m\pi\mathrm{i}}) = (-)^{mn_0}\{\mathrm{e}^{-m\epsilon\pi\mathrm{i}}\,Y_n(z) + 2\mathrm{i}\sin m\epsilon\pi\,\cot\epsilon\pi\,J_n(z)\}. \tag{8·24}$$

The pattern of the zeros of $Y_n(z)$ for general phase ranges of z evidently depends on ϵ, the non-integral part of n. We have considered above the case of integer n; we now deal briefly with the case when n is half an odd integer.

If we put $\epsilon = \frac{1}{2}$ in (8·24), we obtain

$$Y_n(z\,\mathrm{e}^{m\pi\mathrm{i}}) = (-)^{mn_0}\,\mathrm{e}^{-\frac{1}{2}m\pi\mathrm{i}}\,Y_n(z).$$

The zeros of $Y_n(z\,\mathrm{e}^{m\pi\mathrm{i}})$ are thus the same as those of $Y_n(z)$. Because of this and the conjugate property it is sufficient to examine the region $-\frac{1}{2}\pi \leqslant \arg z \leqslant 0$. In this quadrant $Y_n(z)$ has a string of real zeros $y_{n,1}, y_{n,2}, \ldots$, and a number of complex zeros $\overline{\eta}_{n,1}, \overline{\eta}_{n,2}, \ldots$, lying near the lower boundary of **nK** and having the expansion (8·1). The curve on which they lie meets the imaginary z-axis at the point

$$z = n\overline{z}_0 - \tfrac{1}{2}\mathrm{i}(1 - t_0^{-2})^{\frac{1}{2}}\ln 2 + O(n^{-1}) \tag{8·25}$$

(cf. (8·20)), where $\overline{z}_0$ is given by (8·19). The number of such zeros lying to the right of this axis is $[\frac{1}{2}n_0]$; if n_0 is odd there is, in addition, a zero on the imaginary axis.

Figure 13 shows the distribution of zeros over the whole plane. The continuous curves are the boundaries of **nK**. The zeros lie on the real axis outside **nK** and on the broken curves. The latter intersect the imaginary axis at the point $n\overline{z}_0 - \mathrm{i}b$ and its conjugate, where $\overline{z}_0$ and b are given by (8·19) and (8·23). The total number of zeros on each broken curve is $n_0 \equiv n - \frac{1}{2}$.

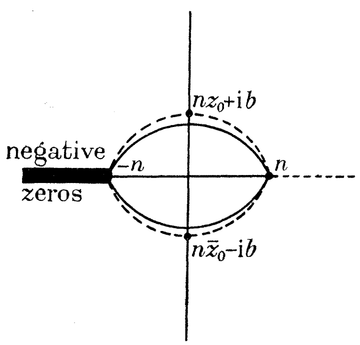

FIGURE 13.

Zeros ---- of $Y_n(z)$, when n is half an odd integer.

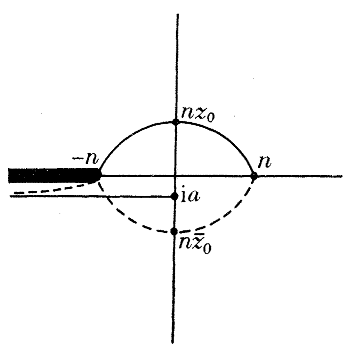

FIGURE 14.

Zeros ---- of $H_n^{(1)}(z)$ in $|\arg z| \leqslant \pi$, when n is an integer.

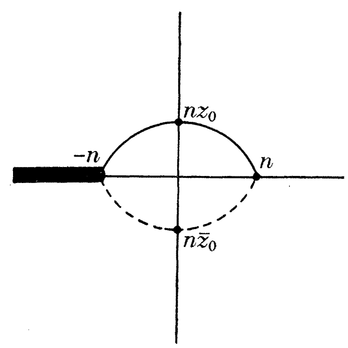

FIGURE 15.

Zeros ---- of $H_n^{(1)}(z)$, when n is half an odd integer.

The same diagram is applicable to the zeros of $Y_n'(z)$, but the number of zeros on each broken curve is now $n_0 + 1 \equiv n + \frac{1}{2}$, and the error term in the asymptotic formulae has to be modified.

As in the case of integer n, it can be shown by contour integration that $Y_n(z)$ and $Y_n(z)$ have no zeros other than those indicated by the asymptotic formulae.

9. Zeros of the Hankel functions and their derivatives

Zeros of $H_n^{(1)}(z)$ and $H_n^{(1)\prime}(z)$ when n is an integer

In this section we extend the analysis of § 8 to the Hankel functions. Only $H_n^{(1)}(z)$ will be considered, the zeros of $H_n^{(2)}(z)$ are the conjugates of those of $H_n^{(1)}(z)$.

If m is any integer we have (Watson 1944, p. 75),

$$H_n^{(1)}(z\,e^{m\pi i}) = e^{-mn\pi i}\,H_n^{(1)}(z) - 2\,e^{-n\pi i}\sin mn\pi\,\operatorname{cosec} n\pi J_n(z). \tag{9·1}$$

Hence if n is an integer

$$H_n^{(1)}(z\,e^{m\pi i}) = (-)^{mn}\{H_n^{(1)}(z) - 2mJ_n(z)\} = (-)^{mn}\,i\{Y_n(z) + (2m-1)\,iJ_n(z)\}. \tag{9·2}$$

Combining this result with (4·24) and (4·26), we obtain

$$H_n^{(1)}(nz\,e^{m\pi i}) \sim (-)^{mn}\,i\left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}}\left\{\frac{\mathrm{Di}_{2m-1}(n^{\frac{2}{3}}\zeta)}{n^{\frac{1}{3}}}\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{n^{2s}} + \frac{\mathrm{Di}'_{2m-1}(n^{\frac{2}{3}}\zeta)}{n^{\frac{5}{3}}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{n^{2s}}\right\}, \tag{9·3}$$

(cf. (8·14)), valid when $|\arg z| < \pi$, where $\mathrm{Di}_m(z)$ is defined by (8·3).

When $m \neq 0$ or 1, $\mathrm{Di}_{2m-1}(z)$ has a string of zeros $d_{2m-1,s}$ near the negative real axis (see § 8). Corresponding to these there is a string of zeros of $H_n^{(1)}(z\,e^{m\pi i})$ lying on a curve whose asymptote is given by

$$\mathscr{I}z = -\tfrac{1}{2}\ln\frac{m}{m-1} + O(n^{-1}) \tag{9·4}$$

(cf. (8·16)). With the aid of Hankel's expansions for $J_n(z)$ and $Y_n(z)$, we can show that the error term $O(n^{-1})$ in this equation is in fact zero. When $m = 0$ or 1, $\mathrm{Di}_{2m-1}(z)$ is a multiple of $\mathrm{Ai}(e^{\pm\frac{2}{3}\pi i}z)$ (see (8·4) and (8·5)), and has no zeros $d_{2m-1,s}$.

Next, if $m \neq 1$, $\mathrm{Di}_{2m-1}(z)$ has a set of zeros $\delta_{2m-1,s}$ near the ray $\arg z = \frac{1}{3}\pi$. Corresponding to these there are zeros of $H_n^{(1)}(z\,e^{m\pi i})$ lying near the lower boundary of **nK**; the curve on which they lie intersects the imaginary z-axis at the point

$$z = n\bar{z}_0 + \tfrac{1}{2}i(1-t_0^{-2})^{\frac{1}{2}}\ln|m-1| + O(n^{-1}) \tag{9·5}$$

(cf. (8·20)). The number of zeros to the right of this axis is $[\frac{1}{2}n+\frac{1}{2}]$ or $[\frac{1}{2}n]$ according as $m \gtrless 1$. If $m = 1$ there are no $\delta_{2m-1,s}$ zeros.

Lastly, if $m \neq 0$, $H_n^{(1)}(z\,e^{m\pi i})$ has a set of zeros lying near the upper boundary of **nK** corresponding to the zeros $\epsilon_{2m-1,s}$ of $\mathrm{Di}_{2m-1}(z)$. The curve of these zeros intersects the imaginary z-axis at the point

$$z = nz_0 - \tfrac{1}{2}i(1-t_0^{-2})^{\frac{1}{2}}\ln|m| + O(n^{-1}), \tag{9·6}$$

and the number of zeros to the right of this axis is $[\frac{1}{2}n]$ or $[\frac{1}{2}n+\frac{1}{2}]$ according as $m \gtrless 0$. If $m = 0$ there are no $\epsilon_{2m-1,s}$ zeros.

The asymptotic distribution of the zeros of $H_n^{(1)}(z)$ in the range

$$(2m-1)\,\pi \leqslant \arg z \leqslant (2m+1)\,\pi$$

may be deduced from these results in a manner similar to that used in § 8 for $Y_n(z)$. For the cases $m \leqslant -1$ and $m \geqslant 1$, the pattern of the zeros is given by figures 10 and 12 respectively, with the quantities a, a', b and b' now given by

$$\left.\begin{aligned} a &= \tfrac{1}{2}\ln\frac{|4m-1|+3}{|4m-1|+1}, & b &= \tfrac{1}{2}(1-t_0^{-2})^{\frac{1}{2}}\ln(|2m-\tfrac{1}{2}|+\tfrac{1}{2}) + O(n^{-1}),\\ a' &= \tfrac{1}{2}\ln\frac{|4m-1|+1}{|4m-1|-1}, & b' &= \tfrac{1}{2}(1-t_0^{-2})^{\frac{1}{2}}\ln(|2m-\tfrac{1}{2}|-\tfrac{1}{2}) + O(n^{-1}),\end{aligned}\right\} \tag{9·7}$$

in place of (8·22). There are n zeros on each of the finite curves.

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Figure 14 corresponds to the case $m = 0$, for which there exist only one finite set and one infinite string of zeros. The former are asymptotically situated on the lower boundary of **nK**, and the asymptote of the latter is

$$\mathscr{I} z = -a = -\tfrac{1}{2}\ln 2 = -0{\cdot}34657\,\ldots.$$

There are n zeros on the finite curve.

Figures 10, 12 and 14 also hold for $H_n^{(1)\prime}(z)$, and on each of the finite curves this function has n zeros also.

It can be verified by contour integration that $H_n^{(1)}(z)$ and $H_n^{(1)\prime}(z)$ have no zeros other than those indicated by the asymptotic formulae.

Zeros of $H_n^{(1)}(z)$ and $H_n^{(1)\prime}(z)$ when n is half an odd integer

If we replace m by $2m$ in (9·1) and substitute $n = n_0 + \frac{1}{2}$, where n_0 is an integer, we obtain

$$H_n^{(1)}(z\,\mathrm{e}^{2m\pi\mathrm{i}}) = (-)^m H_n^{(1)}(z).$$

The zeros of $H_n^{(1)}(z\,\mathrm{e}^{2m\pi\mathrm{i}})$ are thus the same as those of $H_n^{(1)}(z)$.

The expansion of $H_n^{(1)}(nz)$ in the range $|\arg z| < \pi$ is given by (9·3) with $m = 0$. The only zeros in the half-plane $|\arg z| \leqslant \frac{1}{2}\pi$ are those corresponding to $\delta_{-1,s} \equiv \mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,a_s$, and they lie asymptotically close to the lower boundary of **K**. The number of these zeros is $[\frac{1}{2}n_0]$, but if n_0 is odd there is also a zero on the imaginary axis.

If we substitute $m = \pm 1$, $n = n_0 + \frac{1}{2}$ in (9·1), we obtain

$$H_n^{(1)}(z\,\mathrm{e}^{\pm\pi\mathrm{i}}) = \mp(-)^{n_0}\,\mathrm{i}\{H_n^{(1)}(z) - 2J_n(z)\} = \pm(-)^{n_0}\,\mathrm{i}H_n^{(2)}(z) = \pm(-)^{n_0}\,\mathrm{i}\overline{H_n^{(1)}(\bar{z})},$$

and so the zeros of $H_n^{(1)}(z)$ in the quadrants $\frac{1}{2}\pi \leqslant \arg z \leqslant \pi$ and $-\pi \leqslant \arg z \leqslant -\frac{1}{2}\pi$ are the images in the imaginary axis of the zeros in the half-plane $|\arg z| \leqslant \frac{1}{2}\pi$.

Figure 15 illustrates the distribution of zeros over the z-plane. They lie asymptotically close to the lower boundary of **nK** and their total number is $n_0 \equiv n - \frac{1}{2}$. The zeros of $H_n^{(1)\prime}(z)$ are similarly distributed but their total number is $n_0 + 1 \equiv n + \frac{1}{2}$. It is evident from the form of the explicit formulae for $H_{n_0+\frac{1}{2}}^{(1)}$ and $H_{n_0+\frac{1}{2}}^{(1)\prime}$ (see, for example, Watson 1944, p. 80, equation (12)) that there are no zeros other than those indicated by the asymptotic formulae.

10. Tables for the calculation of zeros and associated quantities

For convenient numerical application of the asymptotic series of §§ 7, 8 and 9 we need tables of the coefficients $p_s(\zeta)$, $q_s(\zeta)$, $P_s(\zeta)$, $Q_s(\zeta)$, $\psi(\zeta)$ and $\phi(\zeta)$, and also of Airy function zeros. For real zeros this initial preparation is not prohibitive. The *British Association Mathematical Tables* (1946) give the first fifty zeros of Ai (z) and Ai′ (z) and the first twenty real zeros of Bi (z) and Bi′ (z) to eight decimal places. The writer has prepared comprehensive tables of coefficients which enable the corresponding Bessel function zeros and associated quantities to be computed to ten significant figures when $n \geqslant 5$, and they will be published elsewhere.

For the purpose of evaluating the complex zeros of $Y_n(z)$, $Y_n'(z)$, the Hankel functions and their derivatives in all phase ranges when n is a positive integer, the coefficients $p_s(\zeta)$, $q_s(\zeta)$, …, need to be tabulated over a substantial part of the ζ-plane. Tables would also be required of the zeros of the Airy functions $\mathrm{Di}_m(z)$ and $\mathrm{Di}_m'(z)$, defined by (8·3), for all integer values of m. The preparation of comprehensive tables would be a large undertaking. In this section we give a useful set of skeleton tables which can be used, even for moderately large n, to calculate to three- or four-figure accuracy the smaller complex zeros of Bessel functions and associated quantities in the more important phase ranges.

If the asymptotic formulae of § 8 are curtailed at their first term and the substitution $\psi = 2/z\phi$ (cf. (6·9)) is made, we obtain the following equations, n being a positive integer,

$$\eta = nz + O(n^{-1}), \qquad Y_n'(\eta\,\mathrm{e}^{m\pi\mathrm{i}}) = (-)^{mn+m+1}\,2n^{-\frac{2}{3}}(z\phi)^{-1}\,\mathrm{Di}_{2m}'(\gamma_{2m})\,\{1+O(n^{-2})\}, \quad (10·1)$$

$$\eta' = nz + O(n^{-\frac{1}{3}}), \qquad Y_n(\eta'\,\mathrm{e}^{m\pi\mathrm{i}}) = (-)^{mn}\,n^{-\frac{1}{3}}\phi\,\mathrm{Di}_{2m}(\gamma_{2m}')\,\{1+O(n^{-\frac{4}{3}})\}, \quad (10·2)$$

where γ_{2m}, γ_{2m}' are zeros of $\mathrm{Di}_{2m}(z)$, $\mathrm{Di}_{2m}'(z)$ respectively, and η, η' the corresponding roots of

$$Y_n(\eta\,\mathrm{e}^{m\pi\mathrm{i}}) = 0, \quad Y_n'(\eta'\,\mathrm{e}^{m\pi\mathrm{i}}) = 0.$$

The argument ζ of the functions z and ϕ is $n^{-\frac{2}{3}}\gamma_{2m}$ in (10·1) and $n^{-\frac{2}{3}}\gamma_{2m}'$ in (10·2). The analogous formulae for the Hankel function zeros are (see § 9)

$$\eta = nz + O(n^{-1}), \quad H_n^{(1)\prime}(\eta\,\mathrm{e}^{m\pi\mathrm{i}}) = \mathrm{i}(-)^{mn+m+1}\,2n^{-\frac{2}{3}}(z\phi)^{-1}\,\mathrm{Di}_{2m-1}'(\gamma_{2m-1})\,\{1+O(n^{-2})\}, \quad (10·3)$$

$$\eta' = nz + O(n^{-\frac{1}{3}}), \quad H_n^{(1)}(\eta'\,\mathrm{e}^{m\pi\mathrm{i}}) = \mathrm{i}(-)^{mn}\,n^{-\frac{1}{3}}\phi\,\mathrm{Di}_{2m-1}(\gamma_{2m-1}')\,\{1+O(n^{-\frac{4}{3}})\}, \quad (10·4)$$

where η, η' are now roots of

$$H_n^{(1)}(\eta\,\mathrm{e}^{m\pi\mathrm{i}}) = 0, \qquad H_n^{(1)\prime}(\eta'\,\mathrm{e}^{m\pi\mathrm{i}}) = 0,$$

respectively, and the argument ζ of z and ϕ is $n^{-\frac{2}{3}}\gamma_{2m-1}$ in (10·3) and $n^{-\frac{2}{3}}\gamma_{2m-1}'$ in (10·4). The tables in this section give three-decimal values of γ_m, γ_m', $\mathrm{Di}_m'(\gamma_m)$, $\mathrm{Di}_m(\gamma_m')$, $z(\zeta)$ and $\phi(\zeta)$.

The modulus and phase of the first five complex zeros $\beta_1, \beta_2, \ldots, \beta_5$ of $\mathrm{Bi}(z) = -\mathrm{Di}_0(z)$ and $\beta_1', \beta_2', \ldots, \beta_5'$ of $\mathrm{Bi}'(z)$ are given in table 1, together with the modulus and phase of $\mathrm{Bi}'(\beta_s)$ and $\mathrm{Bi}(\beta_s')$. These zeros lie in the upper half-plane; the corresponding zeros in the lower half-plane are their conjugates.

Tables 2*a*, 2*b* and 2*c* give zeros of the function

$$\mathrm{Di}_2(z) \equiv 2\mathrm{i}\,\mathrm{Ai}(z) - \mathrm{Bi}(z).$$

The first five members of each set $\delta_{2,s}$, $\delta_{2,s}'$, $\epsilon_{2,s}$, $\epsilon_{2,s}'$, $d_{2,s}$ and $d_{2,s}'$, defined in § 8, are given, together with the associated values of Di_2' or Di_2.

Larger zeros of the sets given in tables 1 to 2*c* may readily be found by use of the asymptotic series (8·9) to (8·11). To four-figure accuracy the quantities $T(\lambda)$, $U(\mu)$, $V(\lambda)$ and $W(\mu)$ reduce to a single term when $s \geqslant 6$.

With the aid of tables 1 to 2*c* and the *British Association Mathematical Tables* (1946) giving the real zeros of Ai, Ai′, Bi and Bi′, we may trivially obtain the corresponding zeros and associated values of Di_m and Di_m' for any of the values $m = 0, \pm 1, \pm 2, \pm 3, \pm 5$. From (8·4) and (8·5) it is seen that the zeros of $\mathrm{Di}_{\pm 1}$ are $\mathrm{e}^{\pm\frac{2}{3}\pi\mathrm{i}}$ times those of Ai, the zeros of $\mathrm{Di}_{\pm 3}$ are $\mathrm{e}^{\mp\frac{2}{3}\pi\mathrm{i}}$ times those of Bi, and the zeros of $\mathrm{Di}_{\pm 5}$ are $\mathrm{e}^{\pm\frac{2}{3}\pi\mathrm{i}}$ times those of $\mathrm{Di}_{\mp 2}$, the last relation following from (8·5) with $m = 2, 5$. The same relations hold for the zeros of the derivatives.

We now consider tables of $z(\zeta)$ and $\phi(\zeta)$. In the application of (10·1), (10·2), (10·3) and (10·4) it will be noticed that $\arg\zeta = \arg\gamma_m$ or $\arg\gamma_m'$. In consequence, for the values of m contemplated in tables 1 to 2*c*, $\arg\zeta$ is approximately $\pm\frac{1}{3}\pi$ or π. Thus $z(\zeta)$ and $\phi(\zeta)$ are needed only near the rays $\arg\zeta = \frac{1}{3}\pi$ and $\arg\zeta = \pi$; the values near $\arg\zeta = -\frac{1}{3}\pi$ are obtainable from the former by means of the conjugate property $z(\bar{\zeta}) = \overline{z(\zeta)}$.

Table 3 gives the modulus and phase of the function $z(\zeta)$ for

$$\arg(\zeta e^{-\frac{1}{3}\pi i}) = -0{\cdot}15\,(0{\cdot}05)+0{\cdot}10, \qquad |\zeta| = 0\,(0{\cdot}1)\,l,$$

where l is just greater than the value for which $\arg z = -\frac{1}{2}\pi$. It is linearly interpolable in both directions with a possible error of 2 units in the end figure for the higher values of $|\zeta|$. This table covers all the values of ζ near $\arg\zeta = \frac{1}{3}\pi$ which are required in the application of (10·1) to (10·4) using the zeros of the functions $\mathrm{Di}_m(z)$, $\mathrm{Di}'_m(z)$ $(m = 0, \pm 1, \pm 2, \pm 3, \pm 5)$, with the exception of β'_1, $\delta_{2,1}$, $\delta'_{-2,1}$, $\delta'_{-3,1}$, $\delta_{5,1}$ and $\delta'_{-5,1}$. These exceptions are covered by table 3*a* which gives $z(\zeta)$ for values of $\arg\zeta$ equal to the phases of these particular zeros. This table is linearly interpolable in the $|\zeta|$ direction; the question of interpolation in the $\arg\zeta$ direction does not, of course, arise.

Tables 4 and 4*a* give the values of $\phi(\zeta)$ corresponding to $z(\zeta)$ in tables 3 and 3*a*. They are linearly interpolable.

Tables 5, 5*a*, 6 and 6*a* give, in a similar manner, values of $z(\zeta)$ and $\phi(\zeta)$ near $\arg\zeta = \pi$. They too are linearly interpolable.

Application of the tables: numerical example

When the order n is large and positive, integer or not, we can, with the aid of the tables, calculate the zeros of $Y_n(z)$, $Y'_n(z)$, $H^{(1)}_n(z)$ and $H^{(1)\prime}_n(z)$ in the sector $|\arg z| \leqslant \frac{1}{2}\pi$. When n is a positive integer we can also calculate the zeros in $|\arg z| \leqslant \frac{1}{2}\pi$ of the functions $Y_n(z\,e^{m\pi i})$, $Y'_n(z\,e^{m\pi i})$ $(m = 0, \pm 1)$, and $H^{(1)}_n(z\,e^{m\pi i})$, $H^{(1)\prime}_n(z\,e^{m\pi i})$ $(m = -2\,(1)\,3)$. In terms of general phase ranges this means we can, for integer n, evaluate the zeros of $Y_n(z)$, $Y'_n(z)$ in $|\arg z| \leqslant \frac{3}{2}\pi$, and the zeros of $H^{(1)}_n(z)$, $H^{(1)\prime}_n(z)$ in $-\frac{5}{2}\pi \leqslant \arg z \leqslant \frac{7}{2}\pi$; the corresponding range for $H^{(2)}_n(z)$, $H^{(2)\prime}_n(z)$ is $-\frac{7}{2}\pi \leqslant \arg z \leqslant \frac{5}{2}\pi$.

As an example we evaluate a few complex zeros of $Y_3(z)$. If $n = 3$, then $n^{-\frac{2}{3}} = 0{\cdot}4807$, and using table 1 we obtain

$$|n^{-\frac{2}{3}}\beta_1| = 1{\cdot}132, \qquad \arg(n^{-\frac{2}{3}}\beta_1) = \tfrac{1}{3}\pi + 0{\cdot}095.$$

Entering tables 3 and 4 with $|\zeta| = 1{\cdot}132$, $\arg\zeta = \frac{1}{3}\pi + 0{\cdot}095$, we find by interpolation

$$z = 0{\cdot}783\exp(-0{\cdot}934i), \qquad \phi = 1{\cdot}362\exp(0{\cdot}171i),$$

and so for the corresponding zero $\bar{\eta}_{3,1}$ of $Y_3(z)$ we have approximately (see (10·1))

$$\bar{\eta}_{3,1} = nz = 2{\cdot}349\exp(-0{\cdot}934i), \qquad Y'_3(\bar{\eta}_{3,1}) = 2n^{-\frac{2}{3}}(z\phi)^{-1}\,\mathrm{Bi}'(\beta_1) = 0{\cdot}895\exp(-2{\cdot}879i),$$

the value of $\mathrm{Bi}'(\beta_1)$ being obtained from table 1.

There are two other zeros $\bar{\eta}_{3,2}$ and $\bar{\eta}_{3,3}$, say, near the lower boundary of **nK** (figure 11), but the same procedure with β_2 or β_3 in place of β_1 fails because $|\zeta|$ lies outside the range of table 3. This means that $\bar{\eta}_{3,2}$ and $\bar{\eta}_{3,3}$ lie in the quadrant $-\pi < \arg z < -\frac{1}{2}\pi$. Accordingly, we enter tables 3*a* and 4*a* with the following values obtained from table 2*a*:

$$|\zeta| = |n^{-\frac{2}{3}}\delta_{2,1}| = 0{\cdot}590, \qquad \arg\zeta = \arg\delta_{2,1},$$

and we find that

$$z = 0{\cdot}917\exp(-0{\cdot}472i), \qquad \phi = 1{\cdot}291\exp(0{\cdot}091i).$$

These correspond to a zero of $Y_3(3z\,e^{\pi i})$; the related zero $\eta_{3,3}$ of $Y_3(z)$ is given by

$$\eta_{3,3} = 3\,e^{\pi i}\,z = 2{\cdot}751\exp(2{\cdot}670i),$$

$$Y'_3(\eta_{3,3}) = -2n^{-\frac{2}{3}}(z\phi)^{-1}\,\mathrm{Di}'_2(\delta_{2,1}) = 0{\cdot}696\exp(1{\cdot}480i).$$

This actually lies on the upper branch; the corresponding zero on the lower branch is of course its conjugate. Similarly, using $\delta_{2,2}$ in place of $\delta_{2,1}$ we find

$$\eta_{3,2} = 2{\cdot}202 \exp(1{\cdot}779\mathrm{i}), \qquad Y_3'(\eta_{3,2}) = 0{\cdot}986 \exp(-0{\cdot}958\mathrm{i}).$$

The zeros on the infinite dotted curves of figure 11 may be found in a similar manner. For the first zero on the upper branch, $y_{3,1}^{(1)}$ say, we find that

$$|\zeta| = |n^{-\frac{2}{3}} d_{2,1}| = 1{\cdot}143, \qquad \arg\zeta = \arg d_{2,1} = \pi - 0{\cdot}150 \quad \text{(table } 2c\text{)},$$

$$z = 2{\cdot}138 \exp(-0{\cdot}097\mathrm{i}), \qquad \phi = 1{\cdot}062 \exp(0{\cdot}024\mathrm{i}) \quad \text{(tables 5, 6)},$$

$$\left.\begin{aligned} y_{3,1}^{(1)} &= 3\,\mathrm{e}^{\pi\mathrm{i}} z = 6{\cdot}414 \exp(3{\cdot}045\mathrm{i}) \\ Y_3'(y_{3,1}^{(1)}) &= -2n^{-\frac{2}{3}}(z\phi)^{-1}\,\mathrm{Di}_2'(d_{2,1}) = 0{\cdot}516 \exp(-1{\cdot}534\mathrm{i}) \end{aligned}\right\} \quad \text{(table } 2c\text{)}.$$

For the purpose of comparison accurate values of these zeros of $Y_3(z)$ and the corresponding $Y_3'(z)$ have been computed by solving the equation $Y_3(z) = 0$ by successive approximation, the values of $Y_3(z)$ and $Y_3'(z)$ required being evaluated from their ascending series (Copson 1944, p. 329). The accurate values for the upper zeros are as follows:

$$\eta_{3,1} = 2{\cdot}352 \exp(0{\cdot}932\mathrm{i}), \qquad Y_3'(\eta_{3,1}) = 0{\cdot}895 \exp(2{\cdot}881\mathrm{i}),$$

$$\eta_{3,2} = 2{\cdot}205 \exp(1{\cdot}781\mathrm{i}), \qquad Y_3'(\eta_{3,2}) = 0{\cdot}986 \exp(-0{\cdot}959\mathrm{i}),$$

$$\eta_{3,3} = 2{\cdot}754 \exp(2{\cdot}671\mathrm{i}), \qquad Y_3'(\eta_{3,3}) = 0{\cdot}696 \exp(1{\cdot}478\mathrm{i}),$$

$$y_{3,1}^{(1)} = 6{\cdot}416 \exp(3{\cdot}046\mathrm{i}), \qquad Y_3'(y_{3,1}^{(1)}) = 0{\cdot}516 \exp(-1{\cdot}534\mathrm{i}).$$

The greatest error in the approximate values derived above is about 0·2 %. The discrepancies arise from neglect of $n^{-1}p_1(\zeta)$ and higher terms in (8·1), and behave relatively as $O(n^{-2})$ for large n. We may infer that if $n \geqslant 4$ the values obtained by using formulae (10·1) and (10·3) are accurate to the number of figures provided in the tables. The zeros of the derivatives are a little less accurately determined however, because the relative error in their case is only $O(n^{-\frac{4}{3}})$.

Previous numerical information on the complex zeros of Bessel functions is almost non-existent, save for orders 0 and 1. Complex zeros of Y_0 and Y_1 are given by the National Bureau of Standards Computation Laboratory (1950, pp. 405–406), for example,

$$\eta_{1,1} = -0{\cdot}50274\ 3273 + 0{\cdot}78624\ 3714\mathrm{i}, \qquad \eta_{1,1}' = 0{\cdot}57678\ 5129 + 0{\cdot}90398\ 4792\mathrm{i}.$$

The tables given here yield the approximate values

$$\eta_{1,1} = -0{\cdot}486 + 0{\cdot}786\mathrm{i}, \qquad \eta_{1,1}' = 0{\cdot}604 + 0{\cdot}784\mathrm{i}.$$

It is remarkable that formulae (10·1) and (10·2), which were derived on the assumption that n is large, should give reasonable results even for $n = 1$.

Computation of the tables

The method used for computing tables 1 to $2c$ is not without interest. The asymptotic expansions (8·9) to (8·11) were used to obtain all the quantities required as accurately as the series permitted. The values so obtained were then checked by using ascending series to recalculate the values of Di_m and Di'_m at the zeros. When this check failed, particularly for $s = 1$ and 2 where the asymptotic series yielded insufficient accuracy, new values were calculated by use of Newton's rule and the ascending series evaluated afresh.

The series used for the zeros near the rays $\arg z = \frac{1}{3}\pi$ are based on the fact that Airy functions with argument $z\,\mathrm{e}^{\pm\frac{1}{3}\pi\mathrm{i}}$ can be expressed in terms of Airy functions with argument $-z$; see (8·5). Thus we have

$$\mathrm{Di}_m\,(z\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}) = \tfrac{1}{2}\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\{(m-3)\,\mathrm{i}\,\mathrm{Ai}\,(-z)-(m+1)\,\mathrm{Bi}\,(-z)\}. \tag{10·5}$$

Putting $z = x+h$, and expanding with the aid of Taylor's theorem and the differential equation satisfied by $\mathrm{Ai}\,(-x)$ and $\mathrm{Bi}\,(-x)$, we obtain

$$\mathrm{Di}_m\{(x+h)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\} = \tfrac{1}{2}\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}\left\{F_m(-x)\sum_{r=0}^{\infty}\frac{h^r}{r!}\phi_r + F'_m(-x)\sum_{r=0}^{\infty}\frac{h^r}{r!}\psi_r\right\}, \tag{10·6}$$

$$\mathrm{Di}'_m\{(x+h)\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\} = \tfrac{1}{2}\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\left\{F_m(-x)\sum_{r=0}^{\infty}\frac{h^r}{r!}\phi_{r+1} + F'_m(-x)\sum_{r=0}^{\infty}\frac{h^r}{r!}\psi_{r+1}\right\}, \tag{10·7}$$

where $$F_m(-x) \equiv (m-3)\,\mathrm{i}\,\mathrm{Ai}\,(-x)-(m+1)\,\mathrm{Bi}\,(-x),$$

and ϕ_r, ψ_r are polynomials in x, given by

$$\phi_0 = 1, \quad \phi_1 = 0, \quad \phi_2 = -x, \quad \phi_3 = -1, \quad \phi_4 = x^2,$$
$$\psi_0 = 0, \quad \psi_1 = -1, \quad \psi_2 = 0, \quad \psi_3 = x, \quad \psi_4 = 2,$$

higher members being readily obtainable from the equations

$$\phi_{r+1} = \phi'_r + x\psi_r, \qquad \psi_{r+1} = \psi'_r - \phi_r.$$

In the numerical application of (10·6) and (10·7), the value of x was taken to be a convenient number approximately equal to $\mathscr{R}(\delta_{m,s}\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})$ or $\mathscr{R}(\delta'_{m,s}\,\mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}})$, and the values of $\mathrm{Ai}\,(-x)$, $\mathrm{Bi}\,(-x)$, $\mathrm{Ai}'\,(-x)$ and $\mathrm{Bi}'\,(-x)$ were obtained from the *British Association Mathematical Tables* (1946). Seven terms of each series ensured six-decimal accuracy.

Tables 3 to 6*a* were computed using equations (4·7) and (6·7), the values of σ being found by successive approximation using Newton's rule. All the tables were originally calculated to six decimal places.

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Table 1

s	$e^{-\frac{1}{3}\pi i}\beta_s$ mod.	phase	$Bi'(\beta_s)$ mod.	phase	$e^{-\frac{1}{3}\pi i}\beta'_s$ mod.	phase	$Bi(\beta'_s)$ mod.	phase
1	2·354	0·095	0·993	2·641	1·121	0·331	0·750	0·466
2	4·093	0·042	1·136	−0·513	3·257	0·059	0·592	−2·632
3	5·524	0·027	1·224	2·625	4·824	0·033	0·538	0·515
4	6·789	0·020	1·288	−0·519	6·166	0·023	0·506	−2·624
5	7·946	0·015	1·340	2·622	7·374	0·017	0·484	0·519

Table 2*a*

s	$e^{-\frac{1}{3}\pi i}\delta_{2,s}$ mod.	phase	$Di'_2(\delta_{2,s})$ mod.	phase	$e^{-\frac{1}{3}\pi i}\delta'_{2,s}$ mod.	phase	$Di_2(\delta'_{2,s})$ mod.	phase
1	1·227	0·249	0·857	−2·043	2·312	0·100	0·642	−1·070
2	3·279	0·058	1·076	1·061	4·079	0·042	0·561	2·084
3	4·835	0·033	1·184	−2·086	5·515	0·027	0·520	−1·054
4	6·172	0·023	1·258	1·053	6·783	0·020	0·494	2·090
5	7·378	0·017	1·315	−2·090	7·942	0·015	0·475	−1·051

Table 2*b*

s	$e^{\frac{1}{3}\pi i}\epsilon_{2,s}$ mod.	phase	$Di'_2(\epsilon_{2,s})$ mod.	phase	$e^{\frac{1}{3}\pi i}\epsilon'_{2,s}$ mod.	phase	$Di_2(\epsilon'_{2,s})$ mod.	phase
1	2·344	0·056	1·718	0·537	1·057	0·208	1·307	2·583
2	4·090	0·024	1·967	−2·612	3·251	0·035	1·026	−0·532
3	5·522	0·016	2·119	0·527	4·821	0·019	0·932	2·613
4	6·787	0·011	2·231	−2·615	6·164	0·013	0·877	−0·527
5	7·945	0·009	2·321	0·526	7·373	0·010	0·838	2·615

Table 2*c*

s	$-d_{2,s}$ mod.	phase	$Di'_2(d_{2,s})$ mod.	phase	$-d'_{2,s}$ mod.	phase	$Di_2(d'_{2,s})$ mod.	phase
1	2·377	−0·150	1·219	1·535	1·241	−0·463	0·907	1·657
2	4·101	−0·066	1·392	−1·587	3·270	−0·094	0·725	−1·548
3	5·528	−0·042	1·499	1·560	4·830	−0·052	0·659	1·584
4	6·792	−0·031	1·578	−1·579	6·169	−0·036	0·620	−1·562
5	7·948	−0·025	1·641	1·565	7·376	−0·027	0·593	1·578

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Table 3. $|z|$ and arg z

[$\theta \equiv \arg(\zeta\, e^{-\frac{1}{3}\pi i})$]

θ / $\lvert\zeta\rvert$	−0·15 $\lvert z\rvert$	−0·10 $\lvert z\rvert$	−0·05 $\lvert z\rvert$	0·00 $\lvert z\rvert$	0·05 $\lvert z\rvert$	0·10 $\lvert z\rvert$	−0·15 arg z	−0·10 arg z	−0·05 arg z	0·00 arg z	0·05 arg z	0·10 arg z
0·0	1·000	1·000	1·000	1·000	1·000	1·000	−0·000	−0·000	−0·000	−0·000	−0·000	−0·000
0·1	0·952	0·955	0·958	0·962	0·965	0·969	0·063	0·066	0·068	0·070	0·072	0·073
0·2	0·907	0·913	0·919	0·926	0·933	0·940	0·129	0·134	0·138	0·142	0·145	0·148
0·3	0·864	0·874	0·883	0·893	0·903	0·914	0·197	0·204	0·210	0·216	0·221	0·225
0·4	0·825	0·837	0·850	0·863	0·876	0·890	0·268	0·277	0·285	0·292	0·299	0·304
0·5	0·788	0·803	0·818	0·835	0·852	0·869	−0·342	−0·353	−0·362	−0·371	−0·378	−0·385
0·6	0·753	0·771	0·789	0·809	0·829	0·850	0·418	0·431	0·442	0·452	0·460	0·467
0·7	0·720	0·741	0·763	0·785	0·809	0·833	0·497	0·511	0·524	0·535	0·544	0·551
0·8	0·690	0·713	0·738	0·764	0·791	0·819	0·579	0·594	0·608	0·620	0·630	0·637
0·9	0·662	0·688	0·716	0·745	0·775	0·806	0·663	0·680	0·695	0·707	0·717	0·725
1·0	0·635	0·664	0·695	0·727	0·761	0·796	−0·751	−0·769	−0·785	−0·797	−0·807	−0·814
1·1	0·611	0·643	0·677	0·712	0·750	0·789	0·841	0·860	0·876	0·889	0·898	0·905
1·2	0·588	0·623	0·660	0·699	0·740	0·783	0·935	0·955	0·971	0·984	0·992	0·997
1·3	0·567	0·605	0·645	0·688	0·733	0·780	1·031	1·052	1·068	1·080	1·088	1·091
1·4	0·548	0·589	0·632	0·678	0·728	0·779	1·132	1·153	1·169	1·180	1·186	1·187
1·5	0·530	0·574	0·621	0·671	0·725	0·781	−1·235	−1·257	−1·272	−1·281	−1·286	−1·285
1·6	0·514	0·561	0·612	0·666	0·724	0·786	1·343	1·364	1·378	1·386	1·388	1·384
1·7	0·499	0·549	0·604	0·663	0·727	0·794	1·454	1·475	1·487	1·493	1·492	1·485
1·8	0·485	0·539	0·599	0·663	0·732	0·805	1·570	1·589	1·601	1·603	1·599	1·587
1·9	0·473	—	—	—	—	—	1·689	—	—	—	—	—

Table 3a. $|z|$ and arg z

arg ζ / $\lvert\zeta\rvert$	arg β_1' $(\frac{1}{3}\pi+0{\cdot}33073)$ $\lvert z\rvert$	arg z	arg $\delta_{2,1}$ $(\frac{1}{3}\pi+0{\cdot}24909)$ $\lvert z\rvert$	arg z	arg $\delta'_{-2,1}$ $(\frac{1}{3}\pi-0{\cdot}20756)$ $\lvert z\rvert$	arg z	arg $\delta'_{-3,1}$ $(\frac{1}{3}\pi-0{\cdot}33073)$ $\lvert z\rvert$	arg z	arg $\delta_{5,1}$ $(\frac{1}{3}\pi-0{\cdot}24909)$ $\lvert z\rvert$	arg z	arg $\delta'_{-5,1}$ $(\frac{1}{3}\pi-0{\cdot}46335)$ $\lvert z\rvert$	arg z
0·0	1·000	−0·000	1·000	−0·000	1·000	−0·000	1·000	−0·000	1·000	−0·000	1·000	−0·000
0·1	0·986	0·078	0·980	0·077	0·949	0·060	0·942	0·053	0·946	0·058	0·935	0·045
0·2	0·975	0·158	0·962	0·155	0·900	0·123	0·887	0·109	0·895	0·119	0·874	0·092
0·3	0·966	0·238	0·947	0·235	0·854	0·189	0·835	0·168	0·847	0·182	0·816	0·142
0·4	0·959	0·318	0·934	0·315	0·811	0·257	0·785	0·229	0·802	0·248	0·761	0·195
0·5	0·955	−0·400	0·924	−0·397	0·771	−0·328	0·739	−0·293	0·760	−0·317	0·710	−0·250
0·6	0·954	0·482	0·916	0·480	0·733	0·402	0·695	0·360	0·720	0·389	0·661	0·307
0·7	0·955	0·564	0·910	0·563	0·698	0·478	0·654	0·430	0·682	0·463	0·614	0·368
0·8	0·958	0·647	0·907	0·648	0·664	0·558	0·615	0·503	0·647	0·541	0·571	0·432
0·9	0·965	0·730	0·907	0·733	0·633	0·640	0·579	0·580	0·614	0·621	0·530	0·498
1·0	0·974	−0·813	0·909	−0·819	0·604	−0·726	0·544	−0·659	0·583	−0·705	0·492	−0·568
1·1	0·986	0·897	0·913	0·906	0·577	0·814	0·512	0·742	0·553	0·792	0·456	0·641
1·2	1·001	0·980	0·921	0·993	0·551	0·906	0·482	0·828	0·526	0·883	0·422	0·717
1·3	1·020	1·063	0·931	1·081	0·527	1·002	0·453	0·918	0·500	0·977	0·390	0·797
1·4	1·042	1·146	0·945	1·169	0·505	1·101	0·426	1·011	0·476	1·074	0·360	0·880
1·5	1·068	−1·228	0·963	−1·257	0·484	−1·204	0·401	−1·108	0·454	−1·176	0·332	−0·966
1·6	1·099	1·310	0·984	1·345	0·465	1·310	0·377	1·209	0·433	1·281	0·306	1·056
1·7	1·134	1·391	1·009	1·433	0·447	1·421	0·354	1·315	0·413	1·390	0·282	1·150
1·8	1·175	1·470	1·040	1·521	0·430	1·536	0·333	1·424	0·394	1·504	0·259	1·247
1·9	1·221	1·548	1·076	1·608	0·414	1·655	0·313	1·537	0·376	1·622	0·237	1·347
2·0	1·274	−1·625	—	—	—	—	0·294	−1·655	—	—	0·217	−1·451
2·1	—	—	—	—	—	—	—	—	—	—	0·199	1·558
2·2	—	—	—	—	—	—	—	—	—	—	0·181	1·669

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Table 4. $|\phi|$ and $\arg\phi$

$[\theta \equiv \arg(\zeta\, e^{-\frac{1}{3}\pi i})]$

θ / $\lvert\zeta\rvert$	−0·15 $\lvert\phi\rvert$	−0·10 $\lvert\phi\rvert$	−0·05 $\lvert\phi\rvert$	0·00 $\lvert\phi\rvert$	0·05 $\lvert\phi\rvert$	0·10 $\lvert\phi\rvert$	−0·15 $\arg\phi$	−0·10 $\arg\phi$	−0·05 $\arg\phi$	0·00 $\arg\phi$	0·05 $\arg\phi$	0·10 $\arg\phi$
0·0	1·260	1·260	1·260	1·260	1·260	1·260	0·000	0·000	0·000	0·000	0·000	0·000
0·1	1·272	1·272	1·271	1·270	1·269	1·268	0·012	0·013	0·013	0·014	0·014	0·014
0·2	1·285	1·283	1·282	1·280	1·278	1·276	0·025	0·026	0·027	0·028	0·028	0·029
0·3	1·298	1·295	1·293	1·290	1·287	1·285	0·037	0·039	0·040	0·042	0·043	0·044
0·4	1·311	1·308	1·304	1·301	1·297	1·293	0·050	0·052	0·054	0·056	0·057	0·058
0·5	1·324	1·320	1·316	1·311	1·306	1·302	0·063	0·065	0·067	0·070	0·071	0·073
0·6	1·338	1·333	1·327	1·322	1·316	1·310	0·075	0·078	0·081	0·084	0·086	0·088
0·7	1·352	1·346	1·339	1·333	1·326	1·319	0·088	0·092	0·095	0·098	0·101	0·103
0·8	1·366	1·359	1·352	1·344	1·336	1·328	0·101	0·105	0·109	0·112	0·116	0·119
0·9	1·381	1·373	1·365	1·356	1·347	1·338	0·113	0·118	0·123	0·127	0·131	0·134
1·0	1·396	1·387	1·378	1·368	1·358	1·347	0·126	0·132	0·137	0·141	0·146	0·150
1·1	1·412	1·402	1·392	1·381	1·369	1·357	0·139	0·145	0·151	0·156	0·161	0·166
1·2	1·429	1·418	1·406	1·394	1·381	1·368	0·151	0·158	0·165	0·171	0·177	0·182
1·3	1·447	1·435	1·422	1·408	1·394	1·379	0·164	0·172	0·179	0·186	0·193	0·198
1·4	1·466	1·452	1·438	1·423	1·407	1·391	0·176	0·185	0·194	0·202	0·209	0·216
1·5	1·485	1·471	1·456	1·439	1·421	1·403	0·189	0·199	0·209	0·218	0·226	0·233
1·6	1·507	1·491	1·475	1·456	1·437	1·416	0·201	0·212	0·223	0·234	0·243	0·251
1·7	1·529	1·513	1·495	1·475	1·453	1·430	0·213	0·226	0·238	0·250	0·261	0·270
1·8	1·554	1·537	1·517	1·496	1·472	1·446	0·224	0·239	0·253	0·267	0·279	0·290
1·9	1·580	—	—	—	—	—	0·235	—	—	—	—	—

Table 4*a*. $|\phi|$ and $\arg\phi$

$\arg\zeta$ / $\lvert\zeta\rvert$	$\arg\beta_1'$ $(\frac{1}{3}\pi+0{\cdot}33073)$		$\arg\delta_{2,1}$ $(\frac{1}{3}\pi+0{\cdot}24909)$		$\arg\delta'_{-2,1}$ $(\frac{1}{3}\pi-0{\cdot}20756)$		$\arg\delta'_{-3,1}$ $(\frac{1}{3}\pi-0{\cdot}33073)$		$\arg\delta_{5,1}$ $(\frac{1}{3}\pi-0{\cdot}24909)$		$\arg\delta'_{-5,1}$ $(\frac{1}{3}\pi-0{\cdot}46335)$	
	$\lvert\phi\rvert$	$\arg\phi$	$\lvert\phi\rvert$	$\arg\phi$	$\lvert\phi\rvert$	$\arg\phi$	$\lvert\phi\rvert$	$\arg\phi$	$\lvert\phi\rvert$	$\arg\phi$	$\lvert\phi\rvert$	$\arg\phi$
0·0	1·260	0·000	1·260	0·000	1·260	0·000	1·260	0·000	1·260	0·000	1·260	0·000
0·1	1·264	0·016	1·265	0·015	1·273	0·012	1·275	0·010	1·274	0·011	1·277	0·009
0·2	1·267	0·031	1·271	0·031	1·287	0·024	1·291	0·021	1·288	0·023	1·294	0·018
0·3	1·271	0·047	1·276	0·046	1·301	0·036	1·306	0·032	1·303	0·034	1·311	0·026
0·4	1·275	0·063	1·281	0·062	1·315	0·048	1·322	0·042	1·317	0·046	1·329	0·035
0·5	1·278	0·079	1·287	0·077	1·329	0·060	1·338	0·053	1·332	0·057	1·347	0·044
0·6	1·282	0·095	1·292	0·093	1·344	0·072	1·355	0·063	1·348	0·069	1·365	0·053
0·7	1·286	0·111	1·298	0·109	1·359	0·084	1·372	0·073	1·363	0·080	1·384	0·061
0·8	1·289	0·128	1·303	0·125	1·374	0·096	1·389	0·084	1·379	0·092	1·403	0·070
0·9	1·293	0·144	1·309	0·142	1·390	0·108	1·407	0·094	1·396	0·103	1·422	0·078
1·0	1·296	0·161	1·315	0·158	1·406	0·119	1·425	0·104	1·413	0·114	1·442	0·086
1·1	1·300	0·179	1·321	0·176	1·423	0·131	1·444	0·114	1·431	0·126	1·463	0·094
1·2	1·303	0·197	1·327	0·193	1·441	0·143	1·464	0·123	1·449	0·136	1·483	0·101
1·3	1·307	0·215	1·333	0·211	1·460	0·154	1·484	0·133	1·469	0·147	1·504	0·108
1·4	1·310	0·234	1·339	0·230	1·480	0·166	1·505	0·142	1·489	0·158	1·526	0·115
1·5	1·313	0·253	1·345	0·249	1·500	0·177	1·527	0·150	1·510	0·168	1·548	0·121
1·6	1·316	0·273	1·352	0·269	1·522	0·187	1·549	0·158	1·532	0·177	1·571	0·127
1·7	1·318	0·294	1·358	0·289	1·545	0·198	1·573	0·165	1·556	0·187	1·594	0·132
1·8	1·320	0·315	1·364	0·311	1·570	0·207	1·597	0·172	1·581	0·195	1·617	0·136
1·9	1·322	0·338	1·370	0·334	1·597	0·216	1·622	0·178	1·607	0·203	1·640	0·140
2·0	1·322	0·361	—	—	—	—	1·648	0·182	—	—	1·663	0·143
2·1	—	—	—	—	—	—	—	—	—	—	1·686	0·146
2·2	—	—	—	—	—	—	—	—	—	—	1·709	0·148

TABLE 5. $|z|$ AND $\arg z$

$[\tau \equiv \arg(-\zeta)]$

τ / $\lvert\zeta\rvert$	0·00 $\lvert z\rvert$	−0·05 $\lvert z\rvert$	−0·10 $\lvert z\rvert$	−0·15 $\lvert z\rvert$	0·00 arg z	−0·05 arg z	−0·10 arg z	−0·15 arg z
0·0	1·000	1·000	1·000	1·000	0†	−0·000	−0·000	−0·000
0·1	1·081	1·081	1·081	1·080		0·004	0·008	0·011
0·2	1·166	1·166	1·165	1·164		0·007	0·015	0·022
0·3	1·255	1·255	1·254	1·252		0·011	0·022	0·032
0·4	1·348	1·347	1·346	1·344		0·014	0·028	0·042
0·5	1·444	1·443	1·442	1·439		−0·017	−0·034	−0·051
0·6	1·544	1·543	1·541	1·538		0·020	0·039	0·059
0·7	1·647	1·646	1·644	1·641		0·022	0·045	0·067
0·8	1·754	1·753	1·751	1·747		0·025	0·050	0·074
0·9	1·865	1·864	1·861	1·856		0·027	0·054	0·081
1·0	1·979	1·978	1·975	1·970		−0·029	−0·059	−0·088
1·1	2·097	2·095	2·092	2·086		0·031	0·063	0·094
1·2	2·218	2·216	2·213	2·207		0·033	0·066	0·100
1·3	2·342	2·341	2·337	2·330		0·035	0·070	0·105
1·4	2·470	2·468	2·464	2·457		0·037	0·073	0·110
1·5	2·601	2·599	2·595	2·587		−0·038	−0·077	−0·115
1·6	2·735	2·734	2·729	2·721		0·040	0·079	0·119
1·7	2·873	2·871	2·866	2·858		0·041	0·082	0·123
1·8	3·013	3·012	3·006	2·998		0·042	0·085	0·127
1·9	3·157	3·155	3·150	3·141		0·044	0·087	0·131
2·0	3·304	3·302	3·296	3·287		−0·045	−0·090	−0·135
2·1	3·454	3·452	3·446	3·436		0·046	0·092	0·138
2·2	3·607	3·605	3·599	3·588		0·047	0·094	0·141
2·3	3·763	3·760	3·754	3·744		0·048	0·096	0·144
2·4	3·921	3·919	3·913	3·902		0·049	0·098	0·147
2·5	4·083	4·081	4·074	4·063		−0·050	−0·100	−0·150

τ / $\lvert\zeta\rvert$	0·00 $\lvert z\rvert$	−0·05 $\lvert z\rvert$	−0·05 arg z
2·5	4·08	4·08	−0·050
2·6	4·25	4·25	0·051
2·7	4·41	4·41	0·052
2·8	4·58	4·58	0·052
2·9	4·76	4·76	0·053
3·0	4·93	4·93	−0·054
3·1	5·11	5·11	0·054
3·2	5·29	5·29	0·055
3·3	5·48	5·47	0·056
3·4	5·66	5·66	0·056
3·5	5·85	5·85	−0·057
3·6	6·04	6·04	0·057
3·7	6·24	6·23	0·058
3·8	6·43	6·43	0·058
3·9	6·63	6·63	0·059
4·0	6·83	6·83	−0·059
4·1	7·03	7·03	0·060
4·2	7·24	7·24	0·060
4·3	7·45	7·45	0·060
4·4	7·66	7·66	0·061
4·5	7·87	7·87	−0·061
4·6	8·09	8·08	0·061
4·7	8·30	8·30	0·062
4·8	8·52	8·52	0·062
4·9	8·74	8·74	0·062
5·0	8·97	8·97	−0·063

† arg z is zero in this column.

TABLE 5*a*. $|z|$ AND $\arg z$

arg ζ / $\lvert\zeta\rvert$	arg $d'_{2,1}$ $(\pi-0{\cdot}46335)$		arg $d'_{3,1}$ $(\pi-0{\cdot}33073)$		arg $d'_{5,1}$ $(\pi-0{\cdot}20756)$	
	$\lvert z\rvert$	arg z	$\lvert z\rvert$	arg z	$\lvert z\rvert$	arg z
0·0	1·000	−0·000	1·000	−0·000	1·000	−0·000
0·1	1·073	0·034	1·077	0·025	1·080	0·016
0·2	1·149	0·067	1·157	0·049	1·163	0·031
0·3	1·229	0·098	1·242	0·071	1·250	0·045
0·4	1·313	0·127	1·330	0·092	1·340	0·058
0·5	1·400	−0·154	1·421	−0·111	1·435	−0·070
0·6	1·491	0·180	1·517	0·130	1·533	0·082
0·7	1·586	0·204	1·615	0·147	1·635	0·093
0·8	1·684	0·227	1·718	0·163	1·740	0·103
0·9	1·786	0·248	1·824	0·178	1·849	0·112
1·0	1·891	−0·269	1·934	−0·193	1·961	−0·121
1·1	2·001	0·288	2·047	0·206	2·077	0·130
1·2	2·114	0·306	2·164	0·219	—	—
1·3	2·230	0·323	—	—	—	—

Table 6. $|\phi|$ and $\arg \phi$

$[\tau \equiv \arg(-\zeta)]$

τ	0·00	−0·05	−0·10	−0·15	0·00	−0·05	−0·10	−0·15
$\lvert\zeta\rvert$	$\lvert\phi\rvert$	$\lvert\phi\rvert$	$\lvert\phi\rvert$	$\lvert\phi\rvert$	$\arg\phi$	$\arg\phi$	$\arg\phi$	$\arg\phi$
0·0	1·260	1·260	1·260	1·260	0†	0·000	0·000	0·000
0·1	1·240	1·240	1·240	1·240		0·001	0·002	0·002
0·2	1·221	1·221	1·221	1·221		0·002	0·003	0·005
0·3	1·202	1·202	1·202	1·202		0·002	0·005	0·007
0·4	1·183	1·183	1·184	1·184		0·003	0·006	0·009
0·5	1·165	1·165	1·166	1·166		0·004	0·008	0·011
0·6	1·148	1·148	1·148	1·149		0·005	0·009	0·014
0·7	1·131	1·131	1·131	1·132		0·005	0·010	0·016
0·8	1·114	1·114	1·115	1·115		0·006	0·012	0·018
0·9	1·098	1·098	1·099	1·099		0·007	0·013	0·020
1·0	1·082	1·082	1·083	1·084		0·007	0·014	0·021
1·1	1·067	1·067	1·068	1·068		0·008	0·016	0·023
1·2	1·052	1·052	1·053	1·054		0·008	0·017	0·025
1·3	1·038	1·038	1·038	1·039		0·009	0·018	0·027
1·4	1·024	1·024	1·024	1·025		0·009	0·019	0·028
1·5	1·010	1·010	1·011	1·012		0·010	0·020	0·030
1·6	0·997	0·997	0·998	0·999		0·010	0·021	0·031
1·7	0·984	0·984	0·985	0·986		0·011	0·022	0·033
1·8	0·972	0·972	0·972	0·973		0·011	0·023	0·034
1·9	0·959	0·960	0·960	0·961		0·012	0·024	0·035
2·0	0·948	0·948	0·949	0·949		0·012	0·024	0·037
2·1	0·936	0·937	0·937	0·938		0·013	0·025	0·038
2·2	0·925	0·925	0·926	0·927		0·013	0·026	0·039
2·3	0·914	0·915	0·915	0·916		0·013	0·027	0·040
2·4	0·904	0·904	0·905	0·906		0·014	0·027	0·041
2·5	0·894	0·894	0·895	0·895		0·014	0·028	0·042

τ	0·00	−0·05	−0·05
$\lvert\zeta\rvert$	$\lvert\phi\rvert$	$\lvert\phi\rvert$	$\arg\phi$
2·5	0·894	0·894	0·014
2·6	0·884	0·884	0·014
2·7	0·874	0·874	0·015
2·8	0·865	0·865	0·015
2·9	0·856	0·856	0·015
3·0	0·847	0·847	0·016
3·1	0·838	0·838	0·016
3·2	0·830	0·830	0·016
3·3	0·821	0·822	0·016
3·4	0·813	0·814	0·017
3·5	0·806	0·806	0·017
3·6	0·798	0·798	0·017
3·7	0·791	0·791	0·017
3·8	0·783	0·784	0·017
3·9	0·776	0·776	0·018
4·0	0·769	0·770	0·018
4·1	0·763	0·763	0·018
4·2	0·756	0·756	0·018
4·3	0·750	0·750	0·018
4·4	0·743	0·743	0·018
4·5	0·737	0·737	0·019
4·6	0·731	0·731	0·019
4·7	0·725	0·725	0·019
4·8	0·720	0·720	0·019
4·9	0·714	0·714	0·019
5·0	0·708	0·708	0·019

† $\arg\phi$ is zero in this column.

Table 6a. $|\phi|$ and $\arg \phi$

$\arg\zeta$	$\arg d'_{2,1}$ $(\pi-0{\cdot}46335)$		$\arg d'_{3,1}$ $(\pi-0{\cdot}33073)$		$\arg d'_{5,1}$ $(\pi-0{\cdot}20756)$	
$\lvert\zeta\rvert$	$\lvert\phi\rvert$	$\arg\phi$	$\lvert\phi\rvert$	$\arg\phi$	$\lvert\phi\rvert$	$\arg\phi$
0·0	1·260	0·000	1·260	0·000	1·260	0·000
0·1	1·242	0·007	1·241	0·005	1·241	0·003
0·2	1·225	0·014	1·223	0·010	1·222	0·006
0·3	1·208	0·021	1·205	0·015	1·203	0·010
0·4	1·191	0·028	1·187	0·020	1·185	0·013
0·5	1·174	0·034	1·170	0·025	1·167	0·016
0·6	1·158	0·041	1·153	0·030	1·150	0·019
0·7	1·142	0·047	1·137	0·034	1·133	0·022
0·8	1·127	0·053	1·121	0·038	1·117	0·024
0·9	1·111	0·059	1·105	0·043	1·101	0·027
1·0	1·097	0·065	1·089	0·047	1·085	0·030
1·1	1·082	0·071	1·075	0·051	1·070	0·032
1·2	1·068	0·076	1·060	0·055	—	—
1·3	1·054	0·081	—	—	—	—

Appendix. Some properties of the Airy functions Ai (z) and Bi (z) in the complex plane

Asymptotic expansions

Let us write

$$\xi \equiv \tfrac{2}{3} z^{\frac{3}{2}}, \tag{A 1}$$

$$u_s \equiv \frac{(2s+1)\,(2s+3)\,(2s+5)\ldots(6s-1)}{s!\,(216)^s}, \qquad v_s \equiv -\frac{6s+1}{6s-1}u_s, \tag{A 2}$$

and

$$\left.\begin{aligned} L(\xi) &\equiv \sum_{s=0}^{\infty}\frac{u_s}{\xi^s} = 1+\frac{3\,.\,5}{1!\,216}\frac{1}{\xi}+\frac{5\,.\,7\,.\,9\,.\,11}{2!\,(216)^2}\frac{1}{\xi^2}+\frac{7\,.\,9\,.\,11\,.\,13\,.\,15\,.\,17}{3!\,(216)^3}\frac{1}{\xi^3}+\ldots,\\ M(\xi) &\equiv \sum_{s=0}^{\infty}\frac{v_s}{\xi^s} = 1-\frac{3\,.\,7}{1!\,216}\frac{1}{\xi}-\frac{5\,.\,7\,.\,9\,.\,13}{2!\,(216)^2}\frac{1}{\xi^2}-\frac{7\,.\,9\,.\,11\,.\,13\,.\,15\,.\,19}{3!\,(216)^3}\frac{1}{\xi^3}-\ldots,\end{aligned}\right\} \tag{A 3}$$

$$\left.\begin{aligned} P(\xi) &\equiv \sum_{s=0}^{\infty}(-)^s\frac{u_{2s}}{\xi^{2s}} = 1-\frac{5\,.\,7\,.\,9\,.\,11}{2!\,(216)^2}\frac{1}{\xi^2}+\frac{9\,.\,11\,.\,13\,.\,15\,.\,17\,.\,19\,.\,21\,.\,23}{4!\,(216)^4}\frac{1}{\xi^4}-\ldots,\\ Q(\xi) &\equiv \sum_{s=0}^{\infty}(-)^s\frac{u_{2s+1}}{\xi^{2s+1}} = \frac{3\,.\,5}{1!\,216}\frac{1}{\xi}-\frac{7\,.\,9\,.\,11\,.\,13\,.\,15\,.\,17}{3!\,(216)^3}\frac{1}{\xi^3}+\ldots,\end{aligned}\right\} \tag{A 4}$$

$$\left.\begin{aligned} R(\xi) &\equiv \sum_{s=0}^{\infty}(-)^s\frac{v_{2s}}{\xi^{2s}} = 1+\frac{5\,.\,7\,.\,9\,.\,13}{2!\,(216)^2}\frac{1}{\xi^2}-\frac{9\,.\,11\,.\,13\,.\,15\,.\,17\,.\,19\,.\,21\,.\,25}{4!\,(216)^4}\frac{1}{\xi^4}+\ldots,\\ S(\xi) &\equiv \sum_{s=0}^{\infty}(-)^s\frac{v_{2s+1}}{\xi^{2s+1}} = -\frac{3\,.\,7}{1!\,216}\frac{1}{\xi}+\frac{7\,.\,9\,.\,11\,.\,13\,.\,15\,.\,19}{3!\,(216)^3}\frac{1}{\xi^3}-\ldots.\end{aligned}\right\} \tag{A 5}$$

Then if $|z|$ is large

$$\mathrm{Ai}\,(z) \sim \tfrac{1}{2}\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\,\mathrm{e}^{-\xi}\,L(-\xi), \qquad \mathrm{Ai}'\,(z) \sim -\tfrac{1}{2}\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\,\mathrm{e}^{-\xi}\,M(-\xi), \quad (|\arg z|<\pi), \tag{A 6}$$

$$\left.\begin{aligned}\mathrm{Ai}\,(-z) &\sim \pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\{\cos(\xi-\tfrac{1}{4}\pi)\,P(\xi)+\sin(\xi-\tfrac{1}{4}\pi)\,Q(\xi)\}\\ \mathrm{Ai}'\,(-z) &\sim \pi^{-\frac{1}{2}}z^{\frac{1}{4}}\{\cos(\xi-\tfrac{3}{4}\pi)\,R(\xi)+\sin(\xi-\tfrac{3}{4}\pi)\,S(\xi)\}\end{aligned}\right\} \quad (|\arg z|<\tfrac{2}{3}\pi), \tag{A 7}$$

$$\mathrm{Bi}\,(z) \sim \pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\,\mathrm{e}^{\xi}\,L(\xi), \qquad \mathrm{Bi}'\,(z) \sim \pi^{-\frac{1}{2}}z^{\frac{1}{4}}\,\mathrm{e}^{\xi}\,M(\xi), \quad (|\arg z|<\tfrac{1}{3}\pi), \tag{A 8}$$

$$\left.\begin{aligned}\mathrm{Bi}\,(-z) &\sim \pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\{\cos(\xi+\tfrac{1}{4}\pi)\,P(\xi)+\sin(\xi+\tfrac{1}{4}\pi)\,Q(\xi)\}\\ \mathrm{Bi}'\,(-z) &\sim \pi^{-\frac{1}{2}}z^{\frac{1}{4}}\{\cos(\xi-\tfrac{1}{4}\pi)\,R(\xi)+\sin(\xi-\tfrac{1}{4}\pi)\,S(\xi)\}\end{aligned}\right\} \quad (|\arg z|<\tfrac{2}{3}\pi), \tag{A 9}$$

$$\left.\begin{aligned}\mathrm{Bi}\,(z\,\mathrm{e}^{\pm\frac{1}{3}\pi\mathrm{i}}) &\sim (2/\pi)^{\frac{1}{2}}\,\mathrm{e}^{\pm\frac{1}{6}\pi\mathrm{i}}\,z^{-\frac{1}{4}}\{\cos(\xi-\tfrac{1}{4}\pi\mp\tfrac{1}{2}\mathrm{i}\ln 2)\,P(\xi)+\sin(\xi-\tfrac{1}{4}\pi\mp\tfrac{1}{2}\mathrm{i}\ln 2)\,Q(\xi)\}\\ \mathrm{Bi}'\,(z\,\mathrm{e}^{\pm\frac{1}{3}\pi\mathrm{i}}) &\sim (2/\pi)^{\frac{1}{2}}\,\mathrm{e}^{\mp\frac{1}{6}\pi\mathrm{i}}\,z^{\frac{1}{4}}\{\cos(\xi+\tfrac{1}{4}\pi\mp\tfrac{1}{2}\mathrm{i}\ln 2)\,R(\xi)+\sin(\xi+\tfrac{1}{4}\pi\mp\tfrac{1}{2}\mathrm{i}\ln 2)\,S(\xi)\}\end{aligned}\right\}$$

$$(|\arg z|<\tfrac{2}{3}\pi), \tag{A 10}$$

Expansions (A 6) and (A 7) for Ai and Ai′ may be derived from the well-known asymptotic expansions of Bessel functions of large argument (Watson 1944, chap. 7), using the relations

$$\mathrm{Ai}\,(z) = \frac{z^{\frac{1}{2}}}{\pi\sqrt{3}}K_{\frac{1}{3}}(\xi), \qquad \mathrm{Ai}\,(-z) = \frac{z^{\frac{1}{2}}}{2\sqrt{3}}\{\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\,H^{(1)}_{\frac{1}{3}}(\xi)+\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}\,H^{(2)}_{\frac{1}{3}}(\xi)\},$$

$$\mathrm{Ai}'\,(z) = -\frac{z}{\pi\sqrt{3}}K_{\frac{2}{3}}(\xi), \qquad \mathrm{Ai}'\,(-z) = \frac{z}{2\sqrt{3}}\{\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}\,H^{(1)}_{\frac{2}{3}}(\xi)+\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\,H^{(2)}_{\frac{2}{3}}(\xi)\}.$$

The expansions (A 8), (A 9) and (A 10) for Bi and Bi′ may be derived from (A 6) and (A 7) by means of the relations

$$\mathrm{Bi}\,(z) = \mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\,\mathrm{Ai}\,(\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}\,z)+\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}\,\mathrm{Ai}\,(\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,z), \quad \mathrm{Bi}'\,(z) = \mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}\,\mathrm{Ai}'\,(\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}\,z)+\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}}\,\mathrm{Ai}'\,(\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}\,z).$$

Distribution of zeros

This may be investigated by Lommel's method (Watson 1944, p. 482). Using the differential equation satisfied by Ai we may verify that

$$\frac{\mathrm{d}}{\mathrm{d}z}\{b\,\mathrm{Ai}\,(az)\,\mathrm{Ai}'\,(bz)-a\,\mathrm{Ai}\,(bz)\,\mathrm{Ai}'\,(az)\} = (b^3-a^3)\,z\,\mathrm{Ai}\,(az)\,\mathrm{Ai}\,(bz),$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\{a^2\,\mathrm{Ai}\,(az)\,\mathrm{Ai}'\,(bz)-b^2\,\mathrm{Ai}\,(bz)\,\mathrm{Ai}'\,(az)\} = (a^3-b^3)\,\mathrm{Ai}'\,(az)\,\mathrm{Ai}'\,(bz),$$

where a, b are constants. Hence if $b^3 \neq a^3$

$$\int_0^1 t\,\mathrm{Ai}\,(at)\,\mathrm{Ai}\,(bt)\,\mathrm{d}t = \frac{b\,\mathrm{Ai}\,(a)\,\mathrm{Ai}'\,(b)-a\,\mathrm{Ai}\,(b)\,\mathrm{Ai}'\,(a)}{b^3-a^3}-\frac{b-a}{b^3-a^3}\mathrm{Ai}\,(0)\,\mathrm{Ai}'\,(0),$$

$$\int_0^1 \mathrm{Ai}'\,(at)\,\mathrm{Ai}'\,(bt)\,\mathrm{d}t = \frac{a^2\,\mathrm{Ai}\,(a)\,\mathrm{Ai}'\,(b)-b^2\,\mathrm{Ai}\,(b)\,\mathrm{Ai}'\,(a)}{a^3-b^3}-\frac{a^2-b^2}{a^3-b^3}\mathrm{Ai}\,(0)\,\mathrm{Ai}'\,(0).$$

Suppose now that a is a non-real zero of $\mathrm{Ai}\,(z)$ or $\mathrm{Ai}'\,(z)$, and b is its conjugate $\bar{a}$. Then

$$\int_0^1 t\,\mathrm{Ai}\,(at)\,\mathrm{Ai}\,(\bar{a}t)\,\mathrm{d}t = -\frac{\bar{a}-a}{\bar{a}^3-a^3}\mathrm{Ai}\,(0)\,\mathrm{Ai}'\,(0) = -\frac{1}{r^2}\frac{\sin\theta}{\sin 3\theta}\mathrm{Ai}\,(0)\,\mathrm{Ai}'\,(0), \qquad \text{(A 11)}$$

$$\int_0^1 \mathrm{Ai}'\,(at)\,\mathrm{Ai}'\,(\bar{a}t)\,\mathrm{d}t = -\frac{a^2-\bar{a}^2}{a^3-\bar{a}^3}\mathrm{Ai}\,(0)\,\mathrm{Ai}'\,(0) = -\frac{1}{r}\frac{\sin 2\theta}{\sin 3\theta}\mathrm{Ai}\,(0)\,\mathrm{Ai}'\,(0), \qquad \text{(A 12)}$$

where $a \equiv r\,\mathrm{e}^{\mathrm{i}\theta}$. The integrals in (A 11) and (A 12) are necessarily positive and

$$\mathrm{Ai}\,(0)\,\mathrm{Ai}'\,(0) < 0.$$

In order to avoid a contradiction $\sin\theta/\sin 3\theta$ and $\sin 2\theta/\sin 3\theta$ must each be positive and finite. Thus non-real zeros can only occur in the sector $|\arg z| < \frac{1}{3}\pi$.

Next, we may readily show that for all sufficiently large R there are no zeros in the sector $|\arg z| \leqslant \frac{2}{3}\pi$, $|z| \leqslant R$, by examining the changes in $\arg \mathrm{Ai}\,(z)$ and $\arg \mathrm{Ai}'\,(z)$ as z traverses the boundary; on the curved part we use the asymptotic expressions given by the leading terms in (A 6), and on the rays $\arg z = \pm\frac{2}{3}\pi$ we use the relations (*British Association Mathematical Tables* 1946, p. B 9)

$$\mathrm{Ai}\,(x\,\mathrm{e}^{\pm\frac{2}{3}\pi\mathrm{i}}) = \tfrac{1}{2}\,\mathrm{e}^{\pm\frac{1}{3}\pi\mathrm{i}}\{\mathrm{Ai}\,(x) \mp \mathrm{i}\,\mathrm{Bi}\,(x)\}, \qquad \mathrm{Ai}'\,(x\,\mathrm{e}^{\pm\frac{2}{3}\pi\mathrm{i}}) = \tfrac{1}{2}\,\mathrm{e}^{\mp\frac{1}{3}\pi\mathrm{i}}\{\mathrm{Ai}'\,(x) \mp \mathrm{i}\,\mathrm{Bi}'\,(x)\}.$$

Thus the zeros of $\mathrm{Ai}\,(z)$ *and* $\mathrm{Ai}'\,(z)$ *are all real and negative.*

Equations (A 11) and (A 12) remain valid with Ai replaced by Bi, but in this case $\mathrm{Bi}\,(0)\,\mathrm{Bi}'\,(0) > 0$. Hence $\sin\theta/\sin 3\theta$ and $\sin 2\theta/\sin 3\theta$ must each be negative, and so non-real zeros can only occur in the sectors $\frac{1}{3}\pi < |\arg z| < \frac{1}{2}\pi$. That $\mathrm{Bi}\,(z)$ and $\mathrm{Bi}'\,(z)$ do have zeros in these sectors may be established by examining the changes in $\arg \mathrm{Bi}\,(z)$ and $\arg \mathrm{Bi}'\,(z)$ as z traverses the contour $OABCDEO$ in figure 16. On this diagram AB and DE are arcs of the circle $|z| = R$, where R is arbitrarily large, and B, D are the points $R\,\mathrm{e}^{(\frac{1}{3}\pi\mp\delta)\mathrm{i}}$, δ being an arbitrary number in the range $0 < \delta < \frac{1}{3}\pi$. The curve BCD has the equation

$$\mathscr{I}\{\tfrac{2}{3}z^{\frac{3}{2}}\} = \text{constant} = \tfrac{2}{3}R^{\frac{3}{2}}\cos\tfrac{3}{2}\delta \equiv p, \qquad \text{(A 13)}$$

say; C is thus the point $R(\cos\frac{3}{2}\delta)^{\frac{2}{3}}\,\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$.

On OA we have

$$\underset{OA}{\triangle} \arg \mathrm{Bi}\,(z) = \underset{OA}{\triangle} \arg \mathrm{Bi}'\,(z) = 0. \tag{A 14}$$

On AB and DE we use the asymptotic formulae given by the first terms in (A 8) and (A 9) respectively. We find without difficulty

$$\left.\begin{aligned} &\underset{AB}{\triangle} \arg \mathrm{Bi}\,(z) = -\tfrac{1}{4}(\tfrac{1}{3}\pi-\delta)+p+o(1), \qquad \underset{AB}{\triangle} \arg \mathrm{Bi}'\,(z) = \tfrac{1}{4}(\tfrac{1}{3}\pi-\delta)+p+o(1),\\ &\underset{DE}{\triangle} \arg \mathrm{Bi}\,(z) = -\tfrac{1}{4}(\tfrac{1}{3}\pi-\delta)+p+o(1), \qquad \underset{DE}{\triangle} \arg \mathrm{Bi}'\,(z) = \tfrac{1}{4}(\tfrac{1}{3}\pi-\delta)+p+o(1), \end{aligned}\right\} \tag{A 15}$$

where p is defined by (A 13).

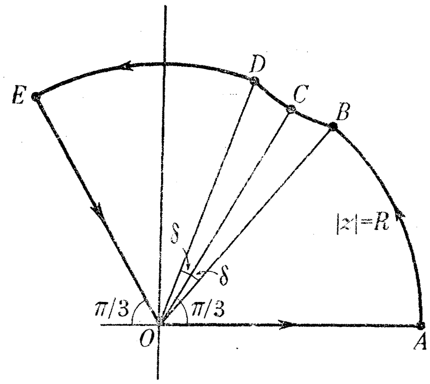

FIGURE 16.

On EO we use the relations

$$\mathrm{Bi}\,(x\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}) = \tfrac{1}{2}\,\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}\{3\,\mathrm{Ai}\,(x)+\mathrm{i}\,\mathrm{Bi}\,(x)\}, \qquad \mathrm{Bi}'\,(x\,\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}) = \tfrac{1}{2}\,\mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}}\{3\,\mathrm{Ai}'\,(x)+\mathrm{i}\,\mathrm{Bi}'\,(x)\}.$$

At $x=0$ we have (*British Association Mathematical Tables* 1946, p. 139)

$$3\,\mathrm{Ai}\,(x)+\mathrm{i}\,\mathrm{Bi}\,(x) = \frac{3^{\frac{1}{3}}}{\Gamma(\frac{2}{3})}\left(1+\frac{\mathrm{i}}{\sqrt{3}}\right), \qquad 3\mathrm{Ai}'\,(x)+\mathrm{i}\,\mathrm{Bi}'\,(x) = \frac{3^{\frac{2}{3}}}{\Gamma(\frac{1}{3})}\left(-1+\frac{\mathrm{i}}{\sqrt{3}}\right),$$

and so

$$\underset{EO}{\triangle} \arg \mathrm{Bi}\,(z) = -\tfrac{1}{3}\pi+o(1), \qquad \underset{EO}{\triangle} \arg \mathrm{Bi}'\,(z) = \tfrac{1}{3}\pi+o(1). \tag{A 16}$$

On BCD we have from (A 10)

$$\mathrm{Bi}\,(z) \sim (2/\pi)^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\,z^{-\frac{1}{4}}\cos\phi, \qquad \mathrm{Bi}'\,(z) = (2/\pi)^{\frac{1}{2}}\,\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}}\,z^{\frac{1}{4}}\sin\phi,$$

where

$$\phi \equiv -\tfrac{2}{3}\mathrm{i}z^{\frac{3}{2}}-\tfrac{1}{4}\pi-\tfrac{1}{2}\mathrm{i}\ln 2.$$

From (A 13) we see that

$$\mathscr{R}\phi = p-\tfrac{1}{4}\pi.$$

Hence if R is chosen so that $p-\frac{1}{4}\pi$ is a multiple of π, $n\pi$ say, then $\cos\phi$ is real and one-signed on the contour, and so

$$\underset{BCD}{\triangle} \arg \mathrm{Bi}\,(z) = -\tfrac{1}{2}\delta+o(1). \tag{A 17}$$

Similarly, if $p+\frac{1}{4}\pi = n\pi$ then

$$\underset{BCD}{\triangle} \arg \mathrm{Bi}'\,(z) = \tfrac{1}{2}\delta+o(1). \tag{A 18}$$

Combining the results (A 14) to (A 18), we find that

$$\underset{OABCDEO}{\triangle} \arg \mathrm{Bi}\,(z) = 2p-\tfrac{1}{2}\pi+o(1) = 2n\pi+o(1),$$

$$\underset{OABCDEO}{\triangle} \arg \mathrm{Bi}'\,(z) = 2p+\tfrac{1}{2}\pi+o(1) = 2n\pi+o(1).$$

Thus if n is a sufficiently large positive integer, the contour $OABCDEO$ has inside it n zeros of $\mathrm{Bi}\,(z)$ *if* $R = \{\frac{3}{2}(n+\frac{1}{4})\,\pi\sec\frac{3}{2}\delta\}^{\frac{2}{3}}$, *and n zeros of* $\mathrm{Bi}'\,(z)$ *if* $R = \{\frac{3}{2}(n-\frac{1}{4})\,\pi\sec\frac{3}{2}\delta\}^{\frac{2}{3}}$.

Asymptotic expansions for the large zeros

Let us write†

$$\left.\begin{aligned} T(\lambda) &\equiv \lambda^{\frac{2}{3}}\left(1+\tfrac{5}{48}\lambda^{-2}-\tfrac{5}{36}\lambda^{-4}+\tfrac{77125}{82944}\lambda^{-6}-\ldots\right), \\ U(\mu) &\equiv \mu^{\frac{2}{3}}\left(1-\tfrac{7}{48}\mu^{-2}+\tfrac{35}{288}\mu^{-4}-\tfrac{181223}{207360}\mu^{-6}+\ldots\right), \\ V(\lambda) &\equiv \frac{\lambda^{\frac{1}{6}}}{\sqrt{\pi}}\left(1+\tfrac{5}{48}\lambda^{-2}-\tfrac{1525}{4608}\lambda^{-4}+\tfrac{2397875}{663552}\lambda^{-6}-\ldots\right), \\ W(\mu) &\equiv \frac{\mu^{-\frac{1}{6}}}{\sqrt{\pi}}\left(1-\tfrac{7}{96}\mu^{-2}+\tfrac{1673}{6144}\mu^{-4}-\tfrac{84394709}{26542080}\mu^{-6}+\ldots\right), \end{aligned}\right\} \tag{A 19}$$

and let the sth negative zeros of Ai (z), Ai′ (z), Bi (z) and Bi′ (z) be denoted by a_s, a'_s, b_s and b'_s respectively. Then if s is large

$$\left.\begin{aligned} & a_s \sim -T(\lambda), && a'_s \sim -U(\mu), \\ & \mathrm{Ai}'(a_s) \sim (-)^{s-1}\,V(\lambda), && \mathrm{Ai}(a'_s) \sim (-)^{s-1}\,W(\mu), \\ \text{where}\quad & \lambda = \tfrac{3}{8}\pi(4s-1), && \mu = \tfrac{3}{8}\pi(4s-3), \end{aligned}\right\} \tag{A 20}$$

and

$$\left.\begin{aligned} & b_s \sim -T(\lambda), && b'_s \sim -U(\mu), \\ & \mathrm{Bi}'(b_s) \sim (-)^{s-1}\,V(\lambda), && \mathrm{Bi}(b'_s) \sim (-)^{s}\,W(\mu), \\ \text{where}\quad & \lambda = \tfrac{3}{8}\pi(4s-3), && \mu = \tfrac{3}{8}\pi(4s-1). \end{aligned}\right\} \tag{A 21}$$

Similarly, if the complex zeros of Bi (z) and Bi′ (z) in the sector $\frac{1}{3}\pi < \arg z < \frac{1}{2}\pi$ arranged in ascending order of modulus are denoted by β_s and β'_s respectively $(s = 1, 2, \ldots)$, then for large s

$$\left.\begin{aligned} & \beta_s \sim \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,T(\lambda), && \beta'_s \sim \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\,U(\mu), \\ & \mathrm{Bi}'(\beta_s) \sim (-)^{s}\,2^{\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}}\,V(\lambda), && \mathrm{Bi}(\beta'_s) \sim (-)^{s-1}\,2^{\frac{1}{2}}\,\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}\,W(\mu), \\ \text{where}\quad & \lambda = \tfrac{3}{8}\{(4s-1)\,\pi+2\mathrm{i}\ln 2\}, && \mu = \tfrac{3}{8}\{(4s-3)\,\pi+2\mathrm{i}\ln 2\}. \end{aligned}\right\} \tag{A 22}$$

The zeros in the sector $-\frac{1}{2}\pi < \arg z < -\frac{1}{3}\pi$ are the conjugates $\bar{\beta}_s$, $\bar{\beta}'_s$.

The expansions (A 20), (A 21) and (A 22) may be derived by reversion of the expansions (A 7), (A 9) and (A 10) respectively. The higher coefficients in formula (A 19) for $V(\lambda)$ and $W(\mu)$ are most conveniently obtained from those of $T(\lambda)$ and $U(\mu)$ by means of the relations (Olver 1950, equation (2·15))

$$\mathrm{Ai}'(a_s) = \pm\left(-\frac{\mathrm{d}a_s}{\mathrm{d}s}\right)^{-\frac{1}{2}}, \qquad \mathrm{Ai}(a'_s) = \pm\left(a'_s\,\frac{\mathrm{d}a'_s}{\mathrm{d}s}\right)^{-\frac{1}{2}}, \tag{A 23}$$

in which s is regarded as a continuous variable.

† Additional terms in the formulae for $T(\lambda)$ and $U(\mu)$ are given in *British Association Mathematical Tables* (1946, p. B48).

References

Bickley, W. G. & Miller, J. C. P. 1945 *Phil. Mag.* **36**, 124–131.

British Association Mathematical Tables 1946 *The Airy integral.* Cambridge University Press.

Cherry, T. M. 1950 *Trans. Amer. Math. Soc.* **68**, 224–257.

Copson, E. T. 1944 *Theory of functions of a complex variable.* Oxford University Press.

Lehmer, D. H. 1945 *Math. Tab., Wash.*, **1**, 133–135.

Miller, J. C. P. 1945 *Math. Tab., Wash.*, **1**, 309–312.

National Bureau of Standards Computation Laboratory 1950 *Table of the Bessel functions* $Y_0(z)$ *and* $Y_1(z)$ *for complex arguments.* New York: Columbia University Press.

Olver, F. W. J. 1950 *Proc. Camb. Phil. Soc.* **46**, 570–580.

Olver, F. W. J. 1951 *Proc. Camb. Phil. Soc.* **47**, 699–712.

Olver, F. W. J. 1952 *Proc. Camb. Phil. Soc.* **48**, 414–427.

Olver, F. W. J. 1954 *Phil. Trans.* A, **247**, 307–327.

Watson, G. N. 1944 *Theory of Bessel functions.* Cambridge University Press.

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